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Gluings Maps, Moduli Spaces of Connections and Donaldson Invariants

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Declaration

The research in this thesis is the original work of the author, with the exception of the attributed sources cited in the text.

Ioannis E.E. Sardis (December 1996)

Summary

In Chapter 1, we follow P. Feehan's iterated conformal blow-ups method, to check that neighbourhoods of boundary points of the compactified moduli space $\bar{\mathcal{M}}_k$ of anti-self-dual connections of charge k which lie on the diagonal of a symmetric sum of copies of the underlying 4-manifold X , are constructible by a gluing process. We then observe that a natural stratification of the associated space of gluing data with respect to the number of points with scale zero, leads to the definition of a space \mathcal{IM}_k which is such that every weakly convergent sequence of \mathcal{M}_k converges into \mathcal{IM}_k with respect to its natural identification topology. In Chapter 3, we consider the moduli space \mathcal{B}_k of all connections of charge k and focus on its C -sequences, namely, sequences of gauge equivalence classes of connections with bounded Yang-Mills energy and functional gradient tending to zero. We employ Taubes' results concerning the limiting behaviour of C -sequences and also certain properties of a general gluing construction, in order to construct a 'limit space' for the C -sequences of \mathcal{B}_k . In Chapter 4, we outline the construction of the μ -map in gauge theory and use the construction of determinant line bundles over $\bar{\mathcal{M}}_k$ associated to certain families of Dirac operators over X , to show that the map μ actually extends over the compactified space. Moreover, we see that the restriction of this extended map to the links of certain low-dimensional strata yields the corresponding μ -map and a symmetric product of the Poincare-dual of a reference homology class. In Chapter 5, we study the restriction of certain products of μ -type cohomology classes to lower strata of the ideal moduli space \mathcal{IM}_k . The formulae emerged from the computation of the associated Kronecker pairings consist of Donaldson polynomials of certain charge and symmetric functions which are defined in terms of the intersection form of the 4-manifold X .

Introduction

In Physics, *Gauge Theories* have been considered as the most promising candidates in an attempt to unify all known fundamental forces in nature within a single coherent theory [32]. The well-known *Maxwell equations* describe electromagnetism in a set of partial differential equations, showing that electricity and magnetism are different manifestations of a single electromagnetic interaction that occurs between moving charged particles. Namely, if E is the *electric field* and B is the *magnetic field*, then Maxwell's equations *in vacuo* are given by $\text{curl}E = dB/dt$, $\text{curl}B = dE/dt$, $\text{div}B = 0$, $\text{div}E = 0$, where *curl* and *div* denote the *rotation* and *divergence* of E and B , respectively. Since these equations are derived from an abelian gauge theory [24], namely from the study of the *Yang-Mills equations*^{†1} over the Minkowski space-time for $U(1)$ -connections on trivial line bundles, the hope has been that non-abelian gauge theories could provide us with further examples of major interest.

For the theoretical physicist, the Yang-Mills equations with gauge group $SU(2)$ yield a non-linear gauge theory [7]. For the mathematician, they have constituted a rich source of information concerning the differential-topology of 4-dimensional manifolds, through the study of certain classes of solutions on principal $SU(2)$ -bundles over the 4-manifold in question. More precisely, one focuses on solutions of the so-called *anti-self-dual (ASD) Yang-Mills equations* that are given by $*F_A = -F_A$. Alternatively, one can recover the *ASD* solutions as the absolute minima of the *Yang-Mills functional* \mathcal{YM} which is defined on the space of connections as the square of the L^2 -Sobolev norm of the curvature [8],[15], observing that the full Yang-Mills equations are the Euler-Lagrange equations of the functional \mathcal{YM} .

^{†1} The *Yang-Mills equations* are given by $*d_A(*F_A) = 0$, where F_A is the *curvature* of the connection A ; d_A is the *exterior derivative* on forms coupled to A and $*$ denotes the *Hodge star operator*.

Let $\mathcal{M}_k = \mathcal{A}_k/\mathcal{G}_k$ be the *moduli space* of equivalence classes of *ASD* connections on a principal $SU(2)$ -bundle P_k , with respect to the action of the group of *gauge transformations* \mathcal{G}_k on the space \mathcal{A}_k of *ASD* connections [8], [15]. The space \mathcal{M}_k was used by S.K. Donaldson to answer the *realization question* for smooth 4-manifolds, namely which symmetric, unimodular, bilinear forms defined over the integers \mathbf{Z} can be realized as intersection forms of some smooth, simply-connected, closed, oriented, Riemannian 4-manifold X . The answer imposed a vast restriction on the number of the above possibilities, stating that if the intersection form Q_X of X is definite, then it is equivalent to the standard diagonal form over \mathbf{Z} [8]. The preceding result combined with Freedman's theorem that every symmetric, unimodular, bilinear form is the intersection form of some topological 4-manifold, imposed a major constraint to the number of 4-manifolds that are smoothable [25]. Let us recall that Freedman's main theorem [16] also gave a complete classification of 4-manifolds (both topological and smooth) up to homeomorphism.

It is worth mentioning that another striking consequence of the combination of the two theorems was the existence of an *exotic* structure on \mathbf{R}^4 , namely that there exist 4-manifolds which are homeomorphic but not diffeomorphic to \mathbf{R}^4 , a phenomenon that occurs only in 4-dimensional topology [15].^{2†} It was later proved by R. Gompf [20] that in fact there exist an infinite family of exotic structures on \mathbf{R}^4 .

The innovation in Donaldson's work was the use of the space of *all* solutions of the given Euler-Lagrange equations rather than a single solution, that in other cases, as for instance in the theory of minimal surfaces suffices to answer the variational problem in question. A conclusion of Donaldson's theory was the existence of an infinite number of simply-connected, smooth 4-manifolds with the same intersection form. According to Freedman's theorem, these 4-manifolds are in the same homeomorphic class. The *Donaldson Polynomials* were introduced to distinguish homeomorphic 4-manifolds which possess different smooth structures and were proved to be insensitive to a change in the metric of the underlying 4-manifold [11]. More precisely, diffeomorphic smooth 4-manifolds possess the same Donaldson polynomials. The polynomials also yielded criteria upon the decomposability of certain classes of smooth 4-manifolds [8]. Despite the development of such a sophisticated theory, the ultimate question in 4-manifold

^{2†} For an extensive discussion, we refer to §1 of [15].

geometry, namely the classification of smooth 4-manifolds up to diffeomorphism, still remains unanswered.

The recent discovery of the *Seiberg-Witten invariants* have changed drastically the 4-manifold scene. More precisely, in an attempt to understand the infrared behaviour of a $N = 2$ supersymmetric Yang-Mills theory in four dimensions, in [49], the authors used rather classical information, as for instance spinors and line bundles, to define 4-manifold invariants by counting solutions of the *monopole equations* for an abelian gauge group. Apart from the crucial fact that the theory is abelian, the beautiful feature of this approach is that the moduli space of solutions is always compact since the bubbling-off appearing in Donaldson theory does not occur here. Consequently, one avoids a great deal of hard, non-linear analysis. Among other things, Seiberg-Witten theory, has led to a proof of the *Thom Conjecture* for the genus of embedded surfaces on \mathbf{CP}^2 and also results on the uniqueness of Einstein metrics. For certain aspects of the relation of Seiberg-Witten theory with Donaldson theory, we refer to §2 of [49].

There is a vast literature on 4-dimensional mathematical gauge theory. We refer to [8], [9], [10] for an extensive geometrical treatment of the subject and to [15], [29] for a fairly analytical one. Brief but juicy accounts of Donaldson's work can be found in [2], [3], [21]. We also refer to [48], [19] for an interpretation of Donaldson theory in terms of a $3 + 1$ Topological Quantum Field Theory, namely a twisted version of a $N = 2$ supersymmetric Yang-Mills theory.

0.1 Gluing constructions in Gauge Theory

Let A_i be connections on principal G -bundles E_i over 4-manifolds X_i , $i = 0, 1$. Roughly speaking, the main idea of a 'gluing construction' in gauge theory is the grafting of A_i , $i = 0, 1$, to each other under rather general hypotheses in order to produce a 'glued' connection $A = A_0 \# A_1$ on a bundle $E = E_0 \# E_1$ over the connected sum $X = X_0 \# X_1$, whose properties would be roughly determined as follows.

(a) The connection A is close to A_i when restricted over each individual summand X_i , $i = 0, 1$, as the scale parameter that measures the size of the 'neck' of X tends to zero.

(b) The *charge* of A , i.e. the *characteristic class* of the connected-sum bundle E , is

equal to the sum of the charges of the connections A_i , $i = 0, 1$.

(c) The connection A inherits certain properties from its ‘predecessors’. For instance, it solves the set of equations that A_i do, or it is ‘fixed’ by the same subgroup of the group of gauge transformations \mathcal{G}_k that fixes A_i , $i = 0, 1$.

In 1982, C.H. Taubes [43] introduced a multiple-gluing operation for solutions of the *ASD* Yang-Mills equations. The problem he considered was the construction of ‘concentrated’ *ASD* Yang-Mills solutions in the sense that the curvature bubbles-off when the neck of the associated connected sum is pinched. Since then numerous versions of the gluing theorems have appeared, varying in their approaches and techniques depending upon the particular application.

Essentially, most of the published constructions [42], [43], [8], use a ‘cut and paste’ operation to graft several *instantons* over the 4-sphere S^4 , i.e. connections whose curvatures are concentrated about the north pole of S^4 , onto a fixed ‘background’ connection A_0 of a reference Riemannian 4-manifold X . One shows then that the parameter family constructed yields a complete model of an explicitly defined open set in the moduli space of X [8].

The range of applications of the gluing construction spans from the derivation of ‘vanishing theorems’ concerning the behaviour of Donaldson Invariants for decomposable 4-manifolds to the description of the ‘ends’ of the compactified *ASD* moduli space $\bar{\mathcal{M}}_k$ [15]. More precisely, the gluing mechanism was used by S. Donaldson in [12] to prove that under certain constraints, the Donaldson polynomials of a 4-manifold which is a connected sum are identically zero, yielding in such a way a restriction to the number of smooth 4-manifolds that are decomposable. The gluing construction also provides us with the key to the analysis of the structure of neighbourhoods of points of \mathcal{M}_k at infinity, namely the boundary of the compactified moduli space can be covered by a finite number of ‘gluing neighbourhoods’ [8], [14], [15].

A fundamental constraint imposed on almost all gluing constructions of the literature is the distinction of the points of the 4-manifold X on which we glue copies of instantons of S^4 , i.e. the requirement that the base points allowed to participate in the process appear with multiplicity 1. Although the extension of the theory in the case that the base points lie in the diagonal of a symmetric product of copies of X seemed to be understood by some authors, no rigorous treatment existed in the literature until

P. Feehan's work [14].

Moreover, the main ingredient missing in the works that discuss the above extension is a proof of the fact that the maps induced by those gluing constructions are in fact diffeomorphisms onto their images. It is worth noting that the assertion that these gluing maps are even homeomorphisms stands perhaps as an act of faith, since no such proof appeared in the publications preceding [14].

In Chapter 1, we discuss briefly the multiple-gluing construction of connections introduced by C. Taubes and we study extensively its generalized version given by P. Feehan [14] in the sense mentioned above. In [14], Taubes' original method is 'iterated' in a way explained more accurately in §1.6, in order to be extended in the case that some of the base points of X coincide.

The key idea of Feehan's construction is the application of a finite number of successive *iterated conformal blow-ups* in order to ensure that even when the points we glue at lie on the diagonal of the symmetric product $Sp^l(X)$, the gluing process still produces an open subset of the *ASD* moduli space \mathcal{M}_k . In this case, instead of gluing several copies of the *standard* 1-instanton over S^4 to a background connection of X_0 , one grafts the so-called *bubble-tree limits* over S^4 produced by the family of iterated conformal blow-ups mentioned above - see §1(c) of [14]. We benefit from this generalized gluing to construct a family of maps $\mathcal{F}_l : \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}) \longrightarrow \mathcal{IM}_{k,l}$, $0 \leq l \leq k$, where the *gluing parameter space* $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})$ consists of the following data.

- (a) The space \mathcal{A}_{k-l} of *ASD* connections on a principal $SU(2)$ bundle P_{k-l} of 2nd Chern class $k-l$ over a closed, oriented, simply-connected, Riemannian 4-manifold X .^{†3}
- (b) The group \mathcal{G}_{k-l} of gauge transformations of P_{k-l} .
- (c) The l -th symmetric product $Sp^l(E_{k-l})$ of the quaternionic vector bundle $E_{k-l} = P_{k-l} \times_{SU(2)} \mathbf{H}$.

Also, $\mathcal{IM}_{k,l}$ denotes the union $\mathcal{IM}_{k,l} = \coprod_{q=0}^l \mathcal{M}_{k-q} \times Sp^q(X)$, $0 \leq l \leq k$. The space $\mathcal{IM}_{k,l}$ given a topology induced by the notion of *weak convergence* of sequences of \mathcal{M}_k to *ideal limits* [44] is often referred to as *ideal moduli space*.

We should stress here that the domains of the gluing maps appearing in this thesis are far more complex than it is indicated by the notation $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})$. Namely, due to the non-canonical nature of the gluing process, the numerous additional choices

^{3†} Henceforth, X will be referred to as *simply-connected*.

of parameters made are fixed at the begining of the particular context.

It is now clear the way in which the results of [14] generalize those of [42]. One can rely on this gluing procedure to construct a map $T_l : \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} D^l(E_{k-l}) \rightarrow \mathcal{M}_k$, where $D^l(E_{k-l}) \subset Sp^l(E_{k-l})$ is defined by

$$D^l(E_{k-l}) = \{[p_1, p_2, \dots, p_l] : p_i, p_j \text{ lie on distinct fibers, } i \neq j, 1 \leq i, j \leq l\}.$$

On the other hand, we use the machinery developed in [14] to define a gluing map $\mathcal{F}_l : \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}) \rightarrow \mathcal{M}_k$, $0 \leq l \leq k$, such that the following diagram commutes.

$$\begin{array}{ccc} \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} D^l(E_{k-l}) & \xrightarrow{T_l} & \mathcal{M}_k \\ \downarrow & & \parallel \\ \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}) & \xrightarrow{\mathcal{F}_l} & \mathcal{M}_k \end{array}$$

The key ingredient of the theory developed in Chapter 1 is that \mathcal{F}_l , $0 \leq l \leq k$, is proved to be a diffeomorphism onto its image and that it identifies neighbourhoods of the stratum $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})$ into the ideal space \mathcal{IM}_k . This implies that every boundary point of the compactified moduli space of *ASD* connections has a neighbourhood that is constructible by gluing.

For our needs, this fact together with the observation that the parameter space $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})$ has a natural stratification with respect to the number of points $(p_1, \dots, p_l) \in Sp^l(E_{k-l})$ with vanishing length, allow us to construct a space \mathcal{JM}_k by connecting the spaces $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})$ and $\mathcal{IM}_{k,l-1}$ along that stratum of $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})$ which is determined by the l -tuple (p_1, \dots, p_l) . Moreover, certain properties of the gluing maps ensure that the space \mathcal{JM}_k with its natural identification topology coincides with the ideal moduli space \mathcal{IM}_k given the topology induced by the notion of weak convergence in the sense explained in Section 1.7.

It is worth mentioning that the method of ‘iterating conformal blow-ups’ has been used in a wide range of applications in geometry and topology. More precisely, the method was initially suggested in [37] in the context of harmonic maps in S^2 , where Sacks and Uhlenbeck studied the existence of minimal 2-spheres. In [41], C. Taubes used an iterated gluing map in order to describe the asymptotical behaviour of sequences of connections with uniformly bounded Yang-Mills energy and functional gradient tending to zero.

However, the closest cousin of the method used in the Yang-Mills context of Chapter 1 is [35], where Parker and Wolfson employed a ‘bubble tree method’ to compactify the moduli space of pseudoholomorphic maps of Riemann surfaces into symplectic manifolds.

0.2 Non-compact variational problems and gluing maps in Gauge Theory

Let \mathcal{B}_k be the moduli space of gauge equivalence classes of *all* connections on a principal $SU(2)$ -bundle P_k of 2nd Chern class k over a *simply-connected* 4-manifold X . Let $\mathcal{YM} : \mathcal{B}_k \rightarrow \mathbb{R}$ be the *Yang-Mills functional* on \mathcal{B}_k defined as the square of the L^2 -Sobolev norm of the curvature. It is known that the moduli space \mathcal{M}_k of solutions of the anti-self-dual Yang-Mills equations for P_k minimizes the functional \mathcal{YM} [8], [25].

The work of S. Donaldson [9], [10] revealed that this special set of critical points of \mathcal{YM} is intimately connected to the topology of the underlying 4-manifold X . Namely, he used the *minimal* manifold $\mathcal{M}_k \subset \mathcal{B}_k$ as a geometrical tool in order to derive until then unknown results concerning the differential-topology of X [8], [15]. Motivated by this phenomenon and since the properties of the full orbit space \mathcal{B}_k are naturally related to the properties of X , one is tempted to investigate whether the non-minimal critical points of \mathcal{YM} on \mathcal{B}_k can give non-trivial information about the topology of X .

The framework one immediately refers to for tackling the above question is Morse theory [31]. However, we note that \mathcal{B}_k is an infinite-dimensional space, therefore non-compact. Consequently, any Morse theory will fail to work for \mathcal{YM} . In [41], C. Taubes suggests a scheme which partly ‘recovers’ Morse theory for \mathcal{YM} on \mathcal{B}_k by analyzing the restrictions of the functional to a countable set of finite-dimensional, non-compact varieties, the so-called *relevant ends* of \mathcal{B}_k with respect to the Yang-Mills functional.

In an attempt to understand the structure of the relevant ends and consequently certain aspects of the topology of \mathcal{B}_k , one examines the limiting behaviour of the so-called *C-sequences*, namely sequences in \mathcal{B}_k with uniformly bounded Yang-Mills energy and functional gradient tending to zero. Let us note here that *C-sequences* in compact manifolds lead to critical points.

In Chapter 2, we prove the existence of a ‘limit space’ for *C-sequences* of \mathcal{B}_k by

completing the non-compact relevant ends of \mathcal{B}_k . Numerous candidates of this kind would probably meet the required convergence-up-to-subsequences criterion. However, an intriguing problem is the construction of a pair (\mathcal{JB}_k, τ) of an infinite-dimensional moduli space \mathcal{JB}_k given a second countable and Hausdorff topology τ , with the following significance.

- (a) Every C -sequence of \mathcal{B}_k converges into (\mathcal{JB}_k, τ) .
- (b) The moduli space \mathcal{JB}_k possesses properties which may be explicitly manipulated for certain purposes of gauge theory.

In order to study the limiting behaviour of C -sequences up to convergence of subsequences, we first employ an *iterative scheme* for a multiple gluing construction that is described in [41]. For our needs, the main result of [41] states that C -sequences of \mathcal{B}_k are within zero L^2_1 -Sobolev distance from sequences of ‘glued’ connections, namely connections which are constructed through a gluing map \mathcal{R}_l similar to the one described in Chapter 1.

We then extend the arguments of [14] in a straightforward way to define a family of embeddings $\mathcal{F}_l : \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}) \rightarrow \mathcal{B}_k$, where \mathcal{A}_{k-l} is the affine space of all connections on a principal $SU(2)$ -bundle P_{k-l} of 2nd Chern class $k-l$ and where the spaces \mathcal{G}_{k-l} and E_{k-l} are as above.

The fact that \mathcal{F}_l is an embedding together with the observation that the parameter space $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})$ has a natural stratification with respect to the number of points $(p_1, \dots, p_l) \in Sp^l(E_{k-l})$ with scale zero, enables us to construct inductively a space \mathcal{JB}_k by connecting the spaces $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})$ and $\mathcal{JB}_{k,l-1}$ along the associated stratum of $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})$ which is determined by the l -tuple (p_1, \dots, p_l) . Then, the observation that by construction the images of the gluing maps \mathcal{R}_l and \mathcal{F}_l differ simply by a diffeomorphism, yields the following theorem.

Theorem 0.2.1 *There is a topological space (\mathcal{JB}_k, τ) such that every C -sequence of \mathcal{B}_k converges to a limit point into (\mathcal{JB}_k, τ) .*

0.3 The μ -map and the topology of moduli spaces of connections

Let \mathcal{B}_k^* be the moduli space of all irreducible connections^{†4} on a principal $SU(2)$ -bundle P_k over a *simply-connected* 4-manifold X . The space \mathcal{B}_k^* is an infinite-dimensional manifold and the space \mathcal{M}_k of equivalence classes of irreducible *ASD* Yang-Mills connections on P_k is a submanifold of \mathcal{B}_k^* .

Although the space \mathcal{A}_k of connections on P_k is topologically trivial,^{†5} this is far from being true for the orbit space \mathcal{B}_k^* . Therefore, a question that immediately poses itself is the computation of the (co)homology of \mathcal{B}_k^* . We fix a base point in X and consider $\tilde{\mathcal{B}}_k$ to be the *base point fibration*, namely an $SO(3)$ -bundle over \mathcal{B}_k^* whose points represent equivalence classes of connections on a bundle which is trivialized over the base point.

In [8], it is shown that if z_1, \dots, z_b is a basis for $H_2(X)$, then the rational cohomology of $\tilde{\mathcal{B}}_k$ is a polynomial algebra on 2-dimensional generators $\mu(z_1), \dots, \mu(z_b)$, where the cohomology class $\mu(z_i)$ is generated by the image of a homomorphism $\mu : H_2(X; \mathbb{Z}) \rightarrow H^2(\tilde{\mathcal{B}}_k; \mathbb{Z})$, the construction of which in the context of *characteristic classes* is outlined in Section 3.2. In terms of classifying spaces the slant product procedure followed to construct μ is presented in Section 2 of [11]. As it is shown in §5.1.2 of [8], the map μ also descends to a homomorphism $\mu : H_2(X; \mathbb{Z}) \rightarrow H^2(\mathcal{B}_k^*; \mathbb{Z})$ and that the rational cohomology of \mathcal{B}_k^* is freely generated as a ring by the image of μ and a 4-dimensional generator β .^{†6}

Let \mathcal{M}_k be the space of equivalence classes of irreducible *ASD* connections of charge k . Since \mathcal{M}_k is naturally embedded into \mathcal{B}_k^* , each class $\mu(z_i)$ can be viewed in $H^2(\mathcal{M}_k)$. Furthermore, as proved in §7.1.4 of [8], the cohomology class $\mu(z)$ extends to a class $\bar{\mu}(z)$ over the compactified moduli space $\bar{\mathcal{M}}_k$ [15] in such a way that its restriction to the l -th link $\bar{\mathcal{M}}_k \cap (\mathcal{M}_{k-l} \times Sp^l(X))$ yields the corresponding class $\mu^{(k-l)}(z) \in H^2(\mathcal{M}_{k-l})$ and the ‘symmetric sum’ $sp^l(PD(z))$ of l copies of the Poincaré dual of z . This fact will be proved crucial for a computation we carry out in Chapter 4 which is related

^{†4} Those connections A whose isotropy group $\Gamma_A = \{g \in \mathcal{G}_k : g(A) = A\}$ coincides with the centre of the structure group G .

^{†5} In the sense that it is contractible as being an affine space.

^{†6} The generator β is the 1st Pontryagin class of the base point fibration.

to Donaldson Polynomial Invariants. The above extension requires the construction of determinant line bundles over $\bar{\mathcal{M}}_k$ associated to certain families of Dirac operators over X and it is presented in Section 3.3.

In fact, the above construction yields a class $\tilde{\mu}(z)$ over the moduli space \mathcal{IM}_k of ideal ASD connections such that its restriction to $\bar{\mathcal{M}}_k$ agrees with the class $\bar{\mu}(z)$. We observe that there are two ways of showing this: to notice that Donaldson's argument actually extends over \mathcal{IM}_k , or alternatively to show that the moduli spaces $\bar{\mathcal{M}}_k$ and \mathcal{IM}_k have isomorphic 2-dimensional cohomology groups since they differ by a set of codimension greater than or equal to 4.

0.4 Donaldson Polynomial Invariants of Smooth 4-manifolds

The classical methods of topology have not succeeded in giving a clear picture of the classification of smooth 4-manifolds up to diffeomorphism. Gauge theory, on the other hand, has managed to partially bridge the gap by assigning to each simply-connected, Riemannian 4-manifold X , a family of finite-dimensional moduli spaces of connections cut out by certain elliptic partial differential equations of first-order, the Yang-Mills equations. Although these equations depend on the Riemannian geometry of X , at the level of homology one finds properties of the moduli spaces that are invariant under the variation of the metric.

The Donaldson Polynomials were introduced in 1990 [9] to shed new light on the differential-topology of the underlying 4-manifold X . They were proved to detect distinct smooth structures on an infinite family of homeomorphic structures of X . For a simply-connected 4-manifold X , let b_X^+ be the dimension of a maximal positive subspace for the intersection form of X [8]. Provided that b_X^+ is greater than 1 and odd, one can define infinitely many Donaldson polynomials which are indexed by an integer k .

Since in most cases these polynomials are very hard to evaluate they remain mysterious. Instead of computing them, one can attempt to understand their structure by establishing constraints or universal relations that they may satisfy. The work of Y. Ruan [36] was a step in this direction: he proved a mod 2 universal constraint and

also derived a structure theorem for Donaldson polynomials of odd Chern class on even manifolds. A major contribution in the study of Donaldson invariants has been made by P. Kronheimer and T. Mrowka [27], [28] concerning the structure of the invariants for 4-manifolds of *simple-type*. A later paper by R. Fintushel and R. Stern [18] studied the invariants for structure group $SO(3)$ under the assumption of simple-type.

In Section 4.2, we give the definition of the Donaldson polynomials using algebraic-topological language. We discuss the fact that they are differential-topological invariants for smooth 4-manifolds and also state two theorems concerning their behaviour for decomposable 4-manifolds as well as for algebraic surfaces. In Section 4.3, we use the natural inclusion of the compactified moduli space $\bar{\mathcal{M}}_k$ of *ASD* connections into the ideal moduli space \mathcal{IM}_k and the properties of the ‘ideal’ map $\tilde{\mu}$ described in Chapter 3, to give a modified description of the Donaldson polynomials using standard algebraic-topological arguments.

More precisely, under the assumption that the moduli space \mathcal{M}_k of *ASD* connections is even-dimensional, $2d$ say, where $d = 4k - \frac{3}{2}(1 + b^+)$, we show that for k within the *stable range* and for a natural homology class $[\mathcal{IM}_k] \in H^{2d}(\mathcal{IM}_k; \mathbf{Z})$, the polynomial functions $\mathcal{D}_k : H^2(X; \mathbf{Z}) \times \dots \times H^2(X; \mathbf{Z}) \rightarrow \mathbf{Z}$ defined by the Kronecker pairing $\mathcal{D}_k(z_1, \dots, z_d) = \langle \tilde{\mu}(z_1) \dots \tilde{\mu}(z_d), [\mathcal{IM}_k] \rangle$ coincide with the Donaldson polynomials.

This alternative description of Donaldson polynomials allows us to consider the restriction of products $\tilde{\mu}(z_1) \dots \tilde{\mu}(z_q)$, $1 \leq q \leq d$, to certain lower strata of \mathcal{IM}_k . Namely, let $j_l : \bar{\mathcal{M}}_{k-l} \times Sp^l(X) \rightarrow \mathcal{IM}_k$, $0 \leq l \leq k$, be the natural inclusion. The evaluation of the Kronecker pairing $\langle (j_l)^*(\tilde{\mu}(z_1) \dots \tilde{\mu}(z_{d-2l})), [\bar{\mathcal{M}}_{k-l} \times Sp^l(X)] \rangle$, $1 \leq l \leq k$, yields a family of formulae which consist of Donaldson polynomials of certain charge and ‘symmetric functions’ which are defined in terms of the intersection form of the 4-manifold X . We also discuss variations of those formulae by weighting the cohomology of \mathcal{IM}_k on homologies of $Sp^l(X)$ other than the top-dimensional one.

In an attempt to cancel certain terms appearing in the derived formulae, we also discuss a conjecture concerning the occasional degeneracy of the above pairings and provide some evidence against its validity.

Chapter 1

On the theory of gluing constructions

1.1 Main Results

Let P_k be a principal $SU(2)$ -bundle over a simply-connected, closed, oriented, Riemannian 4-manifold X . Let \mathcal{A}_k be the space of *ASD* Yang-Mills connections on P_k and \mathcal{G}_k the group of gauge transformations of P_k . With respect to the action of the group \mathcal{G}_k on \mathcal{A}_k , we define the moduli space $\mathcal{M}_k = \mathcal{A}_k/\mathcal{G}_k$ of equivalence classes of *ASD* connections of charge k . A detailed exposition of the fact that \mathcal{M}_k , unlike \mathcal{A}_k , is a finite-dimensional object and that for most purposes it may be assumed to be a smooth manifold except perhaps at a finite collection of singular points associated to reducible connections, can be found in Chapter 4 of [8] and §3 of [15].

Let $\pi : E_k \rightarrow X$ be the projection of the quaternionic vector bundle

$$E_k = P_k \times_{SU(2)} \mathbf{H} ,$$

where the structure group $SU(2)$ acts on the quaternions \mathbf{H} by multiplication on the left and on P_k in the natural way. Let $\mathcal{IM}_{k,l}$ be the set

$$\mathcal{IM}_{k,l} = \coprod_{q=0}^l \mathcal{M}_{k-q} \times Sp^q(X) ,$$

where the q -th symmetric product $Sp^q(X)$ is defined by $Sp^q(X) = \underbrace{X \times \dots \times X}_l / S_l$, where S_l is the permutation group of order l . The space $\mathcal{IM}_{k,l}$ is given a second-countable and Hausdorff topology through the notion of *weak convergence* of sequences

of \mathcal{M}_k to *ideal ASD* connections [8], which one uses to define a natural compactification $\bar{\mathcal{M}}_k$ of \mathcal{M}_k .[†] Let us consider the l -th symmetric product $Sp^l(E_{k-l})$ of the bundle E_{k-l} defined by

$$Sp^l(E_{k-l}) = \frac{\overbrace{E_{k-l} \times E_{k-l} \times \dots \times E_{k-l}}^l}{S_l}.$$

In this chapter, we rely on the gluing constructions of C. Taubes [41], S. Donaldson [8] and P. Feehan [14] concerning the study of connections over connected sums, to show the existence of a gluing map described as follows.

Theorem 1.1.1 *Let X and E_{k-l} be as above. Then, there is a gluing map*

$$\mathcal{F}_l : \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}) \longrightarrow \mathcal{IM}_{k,l},$$

with the following properties: (a) \mathcal{F}_l is defined for all values $1 \leq l \leq k$, (b) \mathcal{F}_l is a diffeomorphism onto its image.

Theorem 1.1.1 stems its importance from the fact that the family of *glued ASD* connections constructed by the map \mathcal{F}_l is proved to describe open neighbourhoods of points on the boundary of the compactified moduli space $\bar{\mathcal{M}}_k$, even if some of the chosen points of the quotient bundle $Sp^l(E_{k-l})$ project onto the diagonal of the symmetric product $Sp^l(X)$.

We essentially follow P. Feehan's arguments [14] concerning the *bubble-tree method*, analyzing the reasons that the standard conformal blow-up techniques probably have to be iterated. We should clarify that Taubes' gluing construction [41], [42] is defined even if some of the points $(p_1, \dots, p_l) \in Sp^l(E_{k-l})$ have multiplicity greater than 1. However, the main ingredient missing for the purpose of this thesis is a proof of the fact that these maps are diffeomorphisms onto their image and that a point on the boundary of the *Uhlenbeck compactification* $\bar{\mathcal{M}}_k$ [15] near the diagonal of $Sp^l(X)$ lies in the image of one of these gluing maps.

We observe that the parameter space $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})$ has a natural filtration with respect to the number of the points in $Sp^l(E_{k-l})$ which have zero norm, namely

$$\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})_l \subset \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})_{l-1} \subset \dots \subset \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})_0,$$

[†] The method is outlined in Section 1.4.

the q -th level of which is defined by

$$\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})_q = \{[A, (p_1, \dots, p_l)] \in \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}) : \text{at most } q-1 \text{ of the points } (p_1, \dots, p_l) \text{ have zero length}\}, \quad 0 \leq q \leq l.$$

In Section 1.7, we consider the stratification of $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})$ corresponding to the above filtration and study the restriction of the gluing map \mathcal{F}_l to the strata $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})_q$, $0 \leq q \leq k-l$. The continuity property of these restrictions enables us to construct inductively a space \mathcal{JM}_k which by construction is such that every weakly convergent sequence of \mathcal{M}_k converges into \mathcal{JM}_k with respect to its natural identification topology.

1.2 Overview of the gluing construction

This section concerns the description of families of *glued* connections over a reference 4-manifold X adapting the gluing methods of S. Donaldson [8], P. Feehan [14] and C. Taubes [41] concerning the parametrization of connections over a connected sum of X with several copies of the 4-sphere S^4 .

Single connected sums: Let us first discuss the construction of *approximately ASD* connections over single connected sums.^{2†} Let P_0 and P_1 be principal $SU(2)$ -bundles of 2nd Chern classes $c_2(P_0) = k_0$ and $c_2(P_1) = k_1$ over Riemannian manifolds X_0 and X_1 , respectively. With respect to the given metrics, let A_0, A_1 be *ASD* connections on P_0 and P_1 , respectively. With these data given, one wants to construct a family of *ASD* connections $\{A = A_0 \# A_1\}$ of *charge* $k_0 + k_1$ on a connected-sum bundle $P = P_0 \# P_1$ over the connected sum $Y = X_0 \# X_1$, such that A is close to A_1 or A_2 when restricted to either X_1 or X_2 .

Let us choose X_0 to be a smooth, simply-connected, closed, oriented, 4-manifold with Riemannian metric g_0 and injectivity radius r_0 and X_1 to be the 4-sphere S^4 with its round metric g_1 of radius 1, induced by the standard metric of \mathbf{R}^5 . Let x_1 be a point in X_0 and n, s denote the north and south pole of S^4 . Let $\mathcal{A}_{k_0}, \mathcal{A}_{k_1}$ be the spaces of *ASD* connections on P_0 and P_1 and $\mathcal{G}_{k_0}, \mathcal{G}_{k_1}$ be the groups of gauge transformations of P_0 and P_1 , respectively.

^{2†} For the convenience of the reader, we have followed closely the notation of [14].

Let us describe briefly the way that the connected sum of X_0 with S^4 is formed. Let $\pi : FrX_0 \rightarrow X_0$ be the principal $SO(4)$ -bundle of oriented orthonormal frames over X_0 . A choice of frame $f \in FrX_0|_{x_1}$ defines a unique geodesic normal coordinate system $\phi_{x_1}^{-1} = \exp_f^{-1} : B_{x_1}(r_0) \rightarrow \mathbf{R}^4$ for the base point $\pi(f) = x_1 \in X_0$. Let also

$$\phi_n^{-1} : U_n = S^4 \setminus \{n\} \rightarrow \mathbf{R}^4, \quad \phi_s^{-1} : U_s = S^4 \setminus \{s\} \rightarrow \mathbf{R}^4$$

be the standard coordinate charts of S^4 . Let $B_{x_1}(r)$ be the geodesic ball in X_0 with centre x_1 and radius r and $B_s(r) = \phi_s(\{x \in \mathbf{R}^4 : |x| < r\})$ be an open ball in S^4 with centre the south pole s . Let $\Omega(x_1, r, R) \subset X_0$ be the open annulus centered at x_1 with inner radius r and outer radius R , i.e. $\Omega(x_1, r, R) = B_{x_1}(R) \setminus \bar{B}_{x_1}(r)$. Similarly, let $\Omega(s, r, R) \subset S^4$ be the open annulus centered at s with radii r and R .

For a fixed large parameter N determined by the construction, let λ_1 be a small positive scale parameter such that $\sqrt{\lambda_1}N \ll 1$. We set

$$X'_0 = X_0 \setminus \bar{B}_{x_1}\left(\frac{\sqrt{\lambda_1}}{N}\right), \quad X''_0 = X_0 \setminus \bar{B}_{x_1}\left(\frac{\sqrt{\lambda_1}}{2}\right), \quad X'''_0 = X_0 \setminus \bar{B}_{x_1}(2N\sqrt{\lambda_1}).$$

Similarly, we define open sets X'_1, X''_1, X'''_1 in $X_1 = S^4$. We consider the annuli

$$\Omega(x_1, \frac{\sqrt{\lambda_1}}{N}, N\sqrt{\lambda_1}) \subset X_0 \quad \text{and} \quad \Omega(s, \frac{\sqrt{\lambda_1}}{N}, N\sqrt{\lambda_1}) \subset X_1.$$

Let $d_1 : \mathbf{R}^4 \rightarrow \mathbf{R}^4$ be a dilation map defined by $d_1(x) = x/\lambda_1$. Let $B_{x_1}(N\sqrt{\lambda_1})$ and $B_{x_1}(2\sqrt{\lambda_1})$ be open balls in X_0 centered at x_1 . We define a diffeomorphism $f_1 : B_{x_1}(\sqrt{\lambda_1}N) \rightarrow X'_1$ by setting $f_1 = \phi_s \circ d_1 \circ \phi_{x_1}^{-1}$. Notice that f_1 identifies the open balls $B_{x_1}(N\sqrt{\lambda_1})$ and $B_{x_1}(2\sqrt{\lambda_1})$ in X_0 with the open sets X'_1 and X''_1 in X_1 and restricts to a diffeomorphism

$$f_1 : \Omega(x_1, \frac{\sqrt{\lambda_1}}{N}, N\sqrt{\lambda_1}) \rightarrow \Omega(s, \frac{\sqrt{\lambda_1}}{N}, N\sqrt{\lambda_1}).$$

One forms the connected sum $X = X_0 \#_{f_1} X_1 \equiv X_0 \cup_{f_1} X_1$ in the familiar way, defining a smooth metric g on X which approximates the metrics on the individual summands and makes f_1 conformal [14]. We outline this method in our description of the gluing operation over multiple connected sums. In this way, (X, g) is conformally equivalent to (X_0, g_0) .

Let A_0 be a g_0 - ASD connection on P_0 over X_0 and A_1 be a g_1 - ASD connection on P_1 over $X_1 = S^4$. We choose $p_0 \in P_0|_{x_1}$ and $p_1 \in P_1|_s$. The point $(A_0, p_0) \in \mathcal{A}_{k_0} \times_{\mathcal{G}_0} P_0|_{x_1}$ determines a local section $\sigma_1(A_0, p_0)$ of P_0 over $B_{x_1}(r_0)$ by the A_0 -parallel transport of p_0 along the radial geodesics through x_1 . This section defines a local trivialization of P_0 over $B_{x_1}(r_0)$, namely $P_0|_{B_{x_1}(r_0)} \simeq B_{x_1}(r_0) \times SU(2)$.

Similarly, the point $(A_1, p_1) \in \mathcal{A}_{k_1} \times_{\mathcal{G}_1} P_1|_s$ defines a section $\sigma_s(A_1, p_1)$ of P_1 over $X_1 \setminus \{n\}$, hence a local trivialization $P_1|_{B_s(r)} \simeq B_s(r) \times SU(2)$.

We replace A_0 by a connection A'_0 which is flat in small neighbourhoods of x_1 . More precisely, we introduce a small parameter $b_1 \geq 4N\sqrt{\lambda_1}$ and we apply a *cutting-off* over the annulus $\Omega(x_1, \frac{b_1}{2}, b_1)$.³ We choose a cut-off function $\psi_0 : X_0 \rightarrow [0, 1]$ defined by⁴

$$\psi_0(y) = \begin{cases} 1, & \text{if } y \in X_0 \setminus B_{x_1}(b_1) \\ 0, & \text{if } y \in B_{x_1}(b_1/2). \end{cases}$$

We define $A'_0 = \psi_0 A_0$ to be a smooth connection on P_0 defined by

$$A'_0 = \begin{cases} A_0, & \text{on } P_0|_{X_0 \setminus B_{x_1}(b_1)} \\ \pi_0^*(\psi_0 \sigma_1^* A_0), & \text{on } P_1|_{B_{x_1}(b_1)}. \end{cases}$$

The connection A'_0 is ‘almost’ ASD , just because of the error terms that these cut-offs introduce and flat on the ball $B_{x_1}(N\sqrt{\lambda_1})$. Similarly, we define a smooth connection A'_1 on P_1 which is almost ASD and flat on the ball $B_s(N\sqrt{\lambda_1}) \subset S^4$.

For the point x_1 , we define the space Gl_{x_1} of *gluing parameters* by

$$Gl_{x_1} \equiv Hom_{SU(2)}(P_0|_{x_1}, P_1|_s) \simeq SO(3)$$

and choose an $SU(2)$ -isomorphism of the fibers $\varrho_1 : (P_0)_{x_1} \rightarrow (P_1)_s$. We use the connections A_i , $i = 0, 1$, over the balls of radius $b_1/2$ to spread out ϱ_1 , in order to obtain a bundle isomorphism $\tilde{\varrho}_1$ over the annuli $\Omega(x_1, \frac{\sqrt{\lambda_1}}{N}, N\sqrt{\lambda_1})$, $\Omega(s, \frac{\sqrt{\lambda_1}}{N}, N\sqrt{\lambda_1})$, which covers the diffeomorphism $f_1 : \Omega(x_1, \frac{\sqrt{\lambda_1}}{N}, N\sqrt{\lambda_1}) \rightarrow \Omega(s, \frac{\sqrt{\lambda_1}}{N}, N\sqrt{\lambda_1})$.

Hence, we obtain that $\sigma_1 \tilde{\varrho}_1 = f_1^* \sigma_s$ on $\Omega(x_1, \frac{\sqrt{\lambda_1}}{N}, N\sqrt{\lambda_1})$. We use $\tilde{\varrho}_1$ to identify the aforementioned trivializations in order to obtain a principal $SU(2)$ -bundle $P = P_0 \# P_1$ of charge $k = k_0 + k_1$ over $Y = X_0 \# S^4$: we simply set $P|_{X'_0} = P_0|_{X'_0}$ and $P|_{X'_1} = P_1|_{X'_1}$. We define a smooth connection $A' = A'_0 \# A'_1$ on P by setting $A' = A'_i$ on the individual summand X_i , $i = 0, 1$.

³† For a detailed account on ‘cutting-off’ techniques, we refer to §4.4.4 of [8].

⁴† For the construction of ψ_0 , we refer to §3.3 of [14].

Remark 1.2.1 For the ‘glued’ connection A , let us use the notation $A(\varrho_1)$ in order to indicate its dependence upon the choice of the fiber-isomorphism ϱ_1 . Let Γ_1, Γ_2 be the isotropy groups of connections $A(\varrho_1), A(\varrho_2)$ which are constructed by the above process for different choices of gluing parameters. Then, $A(\varrho_1)$ is gauge equivalent to $A(\varrho_2)$ if and only if ϱ_1, ϱ_2 lie on the same orbit with respect to the action of $\Gamma = \Gamma_1 \times \Gamma_2$ on the space $G^l_{x_1}$ of gluing parameters - see §7.2.1, §7.2.4 of [8].

Hence, the gluing construction defines a bijection between the set of gluing data described above and a subset of the moduli space $\mathcal{B}_{X,k}$ of gauge equivalence classes of connections of charge k . However, since our objective is to determine a family of smooth connections over X_0 , we use the diffeomorphism f_1 mentioned above to pull back the bundle $P \rightarrow X$ to a bundle $\check{P} \rightarrow X_0$ defined by

$$\check{P} = \begin{cases} P_0|_{X'_0}, & \text{over } X'_0 \\ f_1^* P_1|_{B_{x_1}(N\sqrt{\lambda_1})}, & \text{over } B_{x_1}(N\sqrt{\lambda_1}). \end{cases}$$

The corresponding smooth pull-back connection \check{A}' on \check{P}^\dagger^5 is defined by

$$\check{A}' = \begin{cases} A'_0, & \text{on } \check{P}_0|_{X'_0} \\ f_1^* A_1, & \text{on } \check{P}_0|_{B_{x_1}(N\sqrt{\lambda_1})}. \end{cases}$$

For a local description of \check{A}' over neighbourhoods of x_1 , we refer to §3.3 of [14] and also to §7.2 of [8].

Multiple connected sums: The generalization of the gluing method described above requires the input of a reference simply-connected, Riemannian, 4-manifold X_0 and a collection of summands $X_i = S^4$, $i = 1, \dots, l$. We extend the techniques used in the case of single connected sums in order to describe families of approximately *ASD* connections over the connected sum $X = X_0 \#_{i=1}^l X_i$. We describe the multiple-gluing construction adapting Feehan’s *bubble-tree* presentation. Let us first introduce some terminology that simplifies our description and that it clearly reasons the ‘tree’ structure of the method.

⁵† One checks that \check{P} is defined by transition functions that are constant with respect to the parameter λ_1 - see §3.3 of [14].

1. Terminology: Let $I = (i_1, \dots, i_r)$ be a multi-index of positive integers of length r . Let I_-, I_+ be indices defined by $I_- = (i_1, \dots, i_{r-1})$ and $I_+ = (i_1, \dots, i_{r+1})$. Let \mathcal{I} be an oriented ‘tree’ with a finite set of vertices $\{I\}$ and with *incoming edges* $\{(I_-, I)\}$ and *outgoing edges* $\{(I, I_+)\}$.

2. The gluing data: Let \mathcal{I} be an oriented tree as above. The following data will form the domain of the gluing maps appearing in this chapter.

(a) To a fixed vertex $I \in \mathcal{I}$, we assign a g_I -ASD connection A_I on a principal $SU(2)$ -bundle $P_I \rightarrow X_I$ of 2nd Chern class k_I . If I is the base vertex of length 0, then X_0 is the reference Riemannian 4-manifold of metric g_0 . If I is of positive length, then $X_I = S^4$ with its standard round metric g_I of radius 1.

(b) To each incoming edge $\{(I_-, I)\}$, we assign a tuple $(x_I, v_I, \lambda_I, b_I, \varrho_I, C)$, consisting of (i) a base point $x_I \in X_{I_-}$, (ii) an orthonormal frame $v_I \in FrX_0|_{x_I}$, if $I_- = 0$, (iii) a small neck-size parameter λ_I , (iv) an A_I -cut-off parameter b_I , (v) a gluing parameter $\varrho_I \in SO(3)$, (vi) a 4-tuple C of constants b_0, d_0, λ_0, N , determined by the particular context.

3. The gluing data constraints: There are constraints which are imposed to ensure that gluing regions centred on different base points do not overlap each other and also that no curvature loss occurs over the gluing necks. Namely, one requires:

- (a) the scale parameters λ_I to satisfy $b_I \geq 4N\sqrt{\lambda_I}$, $\lambda_I \leq \lambda_0$,
- (b) the base points $x_I, x_{I'}$ to satisfy
 - (i) $dist_{g_0}(x_I, x_{I'}) > 4(b_I + b_{I'})$, if $I_- = 0$, where $dist_{g_0}$ denotes the geodesic distance on X_0 ,
 - (ii) $|q_I - q_{I'}| > 4(b_I + b_{I'})$, if $I_- > 0$, where the points $q_I \in \mathbf{R}^4$ are such that $x_I = \phi_{I_-}(q_I) \in X_I$,
- (c) the 2nd Chern classes k_I to satisfy $\sum_{I \in \mathcal{I}} k_I = k$ and for at least one $I > 0$ to have $k_I > 0$.

Let us outline the multiple gluing construction.^{†6} For $I_- = 0$, let $\phi_I^{-1} = exp_{v_I}^{-1}$ be a Gaussian coordinate system for $x_I \in X_0$ associated to the frame $v_I \in FrX_0|_{x_I}$ and $B_{x_I}(r)$ be an x_I -centered geodesic ball of radius r . For any $I > 0$, let ϕ_{I_s}, ϕ_{I_n} be

^{†6} For the details we refer to §3.3 of [14], Section 7 of [8] and §6, §7 of [15].

the standard coordinate charts on the 4-sphere X_I and consider the balls $B_{x_I}(r) = \phi_{I_s}(\{x \in \mathbf{R}^4 : |x| < r\})$ and $B_{x_{I_+}}(r) = \phi_{I_n}(\{x \in \mathbf{R}^4 : |x - q_{I_+}| < r\})$.

Let $\Omega(x_I, r, R)$ be the open (r, R) -annulus centered at $x_I \in X_{I_-}$. We consider the open balls $B_{x_I}(N\sqrt{\lambda})$ and the annuli $\Omega(x_I, \frac{\sqrt{\lambda}}{N}, N\sqrt{\lambda})$ in X_{I_-} , $I > 0$. We define the open sets

$$X'_{I_-} = X_{I_-} \setminus \bar{B}_{x_I}(\frac{\sqrt{\lambda_I}}{N}), \quad X''_{I_-} = X_{I_-} \setminus \bar{B}_{x_I}(\frac{\sqrt{\lambda_I}}{2}), \quad X'''_{I_-} = X_{I_-} \setminus \bar{B}_{x_I}(2N\sqrt{\lambda_I}).$$

The diffeomorphisms $f_I : B_{x_I}(\sqrt{\lambda_I}N) \rightarrow X'_I$ defined by $f_I = \phi_{I_s} \circ \phi_{I_+}^{-1}$, identify the open balls $B_{x_I}(N\sqrt{\lambda_I})$ and $B_{x_I}(2N\sqrt{\lambda_I})$ in X_0 with the open sets X'_I and X''_I in X_{I_-} and restrict to diffeomorphisms $f_I : \Omega(x_I, \frac{\sqrt{\lambda_I}}{N}, N\sqrt{\lambda_I}) \rightarrow \Omega(I_s, \frac{\sqrt{\lambda_I}}{N}, N\sqrt{\lambda_I})$. One then forms the connected sum $X = X_0 \cup_{f_I, I \in \mathcal{I}} X_I = \#_{I \in \mathcal{I}} X_I$ in the familiar way.

In Remark 1.2.3, we comment on the definition of a smooth metric g on X which approximates the metrics g_I on the individual summands X_I and makes f_I conformal, i.e. makes (X, g) conformally equivalent to (X_0, g_0) .

Let A_{I_-} be a g_I -ASD connection on P_I over X_I . A point $(A_{I_-}, p_{I_-}) \in \mathcal{A}_{k_{I_-}} \times_{g_{I_-}} P_{I_-}|_{x_I}$ determines a local section σ_I of P_{I_-} over $B_{x_I}(r_0)$ by the A_I -parallel transport of p_I along the radial geodesics through x_I . This section defines a local trivialization of P_{I_-} over $B_{x_I}(r_0)$, namely $P_{I_-}|_{B_{x_I}(r_0)} \simeq B_{x_I}(r_0) \times SU(2)$.

Similarly, a point $(A_I, p_I) \in \mathcal{A}_{k_I} \times_{g_I} P_I|_{I_s}$ defines a section σ_{I_s} of P_I over $X_I \setminus \{I_n\}$ and consequently a local trivialization $P_I|_{B_{I_s}(r)} \simeq B_{I_s}(r) \times SU(2)$.

We replace the connections A_I by connections A'_I which are flat in neighbourhoods of x_I , by performing a cutting-off over the annulus $\Omega(x_I, \frac{b_I}{2}, b_I)$: we choose a cut-off function $\psi_I : X_I \rightarrow [0, 1]$ defined by⁷

$$\psi_I = \begin{cases} (\phi_{I_s}^{-1})^* \psi_{b_I} \Pi_{I_+} (\phi_{I_+}^{-1})^* \psi_{I_+}, & \text{on } X_I \\ \Pi_{I_+} (\phi_{I_+}^{-1})^* \psi_{I_+}, & \text{on } X_0 \end{cases}$$

The connections $A'_{I_-} = \psi_{I_-} A_0$, $A'_I = \psi_I A_0$ are smooth, almost ASD and flat on the balls $B_{x_I}(2N\sqrt{\lambda_I})$, $B_{I_s}(2N\sqrt{\lambda_I})$, respectively. We choose an $SU(2)$ -isomorphism $\varrho_I : (P_{I_-})_{x_I} \rightarrow (P_I)_{I_s}$ of the fibers and use the connections A'_{I_-}, A'_I to spread out ϱ_I .

In this way, we obtain a bundle isomorphism $\tilde{\varrho}_I$ over the annuli $\Omega(x_I, \frac{\sqrt{\lambda_I}}{N}, N\sqrt{\lambda_I})$, $\Omega(I_s, \frac{\sqrt{\lambda_I}}{N}, N\sqrt{\lambda_I})$ which covers the diffeomorphism

⁷† For the construction of ψ_I , we refer to §3.3 of [14].

$$f_I : \Omega(x_I, \frac{\sqrt{\lambda_I}}{N}, N\sqrt{\lambda_I}) \rightarrow \Omega(Is, \frac{\sqrt{\lambda_I}}{N}, N\sqrt{\lambda_i}) .$$

We use $\tilde{\varrho}_I$ to identify the above trivializations and so obtain a principal $SU(2)$ -bundle $P = \#_{I \in \mathcal{I}} P_I$ of charge k over X .^{†8} We define a smooth connection A' on \mathcal{P} by patching together the cut-off connections A'_I .^{†9} Hence, we finally determine a family of smooth connections over X_0 on the pull-back $\tilde{P} \rightarrow X_0$ of $P \rightarrow X$ through the conformal maps f_I .

Remark 1.2.2 As it should be already clear, gluing is a highly non-canonical construction since it depends on the choice of numerous parameters as base points, orthonormal frames, local trivializations, scale parameters and cut-off functions, to mention just a few of them. In [8], [14], [41], the authors employ analytical estimates to prove that certain choices of those parameters determine a constant C such that

$$\|A_I - A'_I\|_{L^p(X_I, g_I)} \leq \frac{C}{\bar{b}^{4/p+1}} \quad \text{and} \quad \|F^+(A'_I)\|_{L^p(X_I, g_I)} \leq \frac{C}{\bar{b}^{4/p}},$$

where $\bar{b} = \max_{I \in \mathcal{I}} b_I$.

Remark 1.2.3 A main ingredient in the gluing constructions over single and multiple connected sums is the definition of a conformal metric g on $X = \#_{I \in \mathcal{I}} X_I$. This requires the replacement of the standard metric g_I on each 4-sphere X_I by a quasi-conformally equivalent metric \bar{g}_I with respect to which the identification maps f_I are conformal. The metric g on X in the conformal class $[g_0] = [g]$ is defined in terms of g_0 and \bar{g}_I . For the construction of g as well as for a comparison of the corresponding L^p estimates, we refer to [10] and §3.5 of [14].

1.2.1 Gluing maps

Let \mathcal{D} be a parameter space of gluing data which is described below. We use the gluing process presented above to construct gluing maps $\mathcal{G} : \mathcal{D} \rightarrow \mathcal{B}_{X_0, k}^*$, certain properties of which we study in later sections.

^{†8} One checks that the transition functions of P are constant with respect to the scale λ_I .

^{†9} For purposes of completion we note that one uses Aronszajn's continuation principle for the forms which set the Laplacian equal to zero, to show that the connection A' is irreducible.

Let us first describe the parameter space \mathcal{D} of gluing data: let A_I be a g_I - ASD connection over X_I with isotropy group Γ_{A_I} and $H_{A_I}^2 = 0$.^{†10} For fixed base points $\{x_I\}$, scales $\{\lambda_I\}$ and gluing parameters $\{\varrho_I\}$, the gluing construction yields a family of almost ASD connections over the connected sum $X = \#_{I \in \mathcal{I}} X_I$.

A larger family of glued connections is obtained by allowing the base points and scales to ‘move’ in the disjoint balls $B_{x_I}(r_0)$ and the interval $(0, \lambda_0)$, respectively. We enlarge the glued family of connections even more using the free action of the isotropy group $\Gamma = \Gamma_0 \prod_{I \in \mathcal{I}} \Gamma_{A_I}$ on the space Gl_{x_I} of gluing parameters.

Hence, having the connections A_I fixed, the space \mathcal{D} of gluing parameters is defined to be the quotient space

$$\mathcal{D} = \frac{Gl_{x_I} \times B_{x_I}(r_0) \times (0, \lambda_0)}{\Gamma_0 \prod_{I \in \mathcal{I}} \Gamma_{A_I}}.$$

Therefore, one defines a map $G : \mathcal{D} \longrightarrow \mathcal{B}_{X,k}^*$ by setting $G(A_0, A_I, z) = [A'(z)]$, $z \in \mathcal{D}$. In the familiar way, we use the conformal diffeomorphisms f_I to define an almost g_0 - ASD connection \check{A}' on the pull-back of the bundle $\mathcal{P} \rightarrow X$. The process defines a smooth map $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{B}_{X_0,k}^*$.

Proposition 5.10 of [14] proves that the gluing maps G and \mathcal{G} are smooth submersions onto their images. The essence of this chapter relies on the fact that G and \mathcal{G} restricted into a smaller parameter space of gluing data yield smooth embeddings. The characteristic of the above parameter space is that for $I > 0$, the chosen g_I - ASD connections of $X_I = S^4$ are properly *centred* in the sense of §1.3.1.

Remark 1.2.4 In §3.3 – §3.5 of [14] and §7.2.4 of [8] it is proved that for given $p \leq 1$, there is a constant C such that for any $z \in \mathcal{D}$ it holds that

$$\|F^+(\check{A}')\|_{L^p(X_I, g_I)} \leq \frac{C(z)}{\bar{b}^{4/p}}.$$

We therefore assert that proper choices of gluing data make the self-dual part of the curvature of \check{A}' over X_0 sufficiently small. As we discuss in Section 1.6, this fact together with an application of the *contraction mapping principle* produce a g_0 - ASD connection $\check{A}' + \check{a}$, where $\check{a} \in \Omega^1(X_0, \text{ad} P_0)$ is unique in the sense explained in the same section and such that its L^4 -Sobolev norm is controlled by the construction.

^{†10} If d_{A_I} denotes the corresponding exterior covariant derivative and $d_{A_I}^+$ is the g_I -self-dual part of d_{A_I} , then $H_{A_I}^2 = \text{coker } d_{A_I}^+$.

1.3 Connections on S^4

The description of *ASD* $SU(2)$ -connections over the 4-sphere S^4 , also called *instantons*, was first given in [4] using the so-called *ADHM* method. Detailed expositions of the *ADHM* construction can also be found in [1] and [8]. For the purposes of this thesis, we need to employ a class of *centred* k -instantons on S^4 described in §1.3.1.

One-instantons on S^4 : An 1-instanton I on S^4 can be assigned a pair (b, r) , the *scale* $r \in \mathbf{R}^+$ determining the distribution of the curvature of I around the *centre* $b \in S^4$. Moreover, the parameters r and b are acted on transitively by the translations and dilations of \mathbf{R}^4 - see [1]. More precisely, the construction of instantons over S^4 is simplified from the fact that we can instead construct *ASD* connections, A_a say, on the Euclidean space \mathbf{R}^4 . The conformal invariance of both the *ASD* Yang-Mills equations and the energy integral enables us then to view A_a as *ASD* connections over $S^4 \setminus \infty$. Finally, according to Uhlenbeck's *removable singularities theorem* [44], these connections can be extended smoothly over S^4 .

Throughout, we identify \mathbf{R}^4 with the space \mathbf{H} of quaternions. An $SU(2)$ -connection on \mathbf{R}^4 will be a differential form $A(x) = \sum_{i=1}^4 A_i(x) dx^i$, $x \in \mathbf{H}$, with values in the imaginary quaternions.^{†11} Therefore, the expression

$$I(x) = \text{Im} \left\{ \frac{\bar{x} dx}{1 + |x|^2} \right\} = \frac{1}{2} \left\{ \frac{\bar{x} dx - d\bar{x} x}{1 + |x|^2} \right\} \quad (1.1)$$

represents an $SU(2)$ -connection whose curvature is given by the formula

$$F_I(x) = \frac{d\bar{x} \wedge dx}{(1 + |x|^2)^2}, \quad x \in \mathbf{H}.$$

A straightforward computation shows that the coefficients of i, j, k in the expression of $d\bar{x} \wedge dx$ form a basis for the anti-self-dual 2-forms. Therefore, $I(x)$ is an *ASD* $SU(2)$ -connection, henceforth called the *standard 1-instanton* I with centre 0 and scale 1. Let now $T_{r,b} : \mathbf{H} \rightarrow \mathbf{H}$ be linear transformations defined by $T_{r,b}(x) = r(x - b)$, $x, b \in \mathbf{H}$, $r \in \mathbf{R}^+$. One can check that the pull-back connection $T_{r,b}^* I$ is given by

$$I_{r,b} \equiv T_{r,b}^* I = \text{Im} \left\{ \frac{r^2 (\bar{x} - \bar{b}) dx}{1 + r^2 |x - b|^2} \right\},$$

^{†11} Since $SU(2)$ is isomorphic to the group $Sp(1)$ of unit quaternions, the Lie algebra $\mathfrak{su}(2)$ gets identified with the imaginary quaternions $\mathfrak{sp}(1)$.

and its curvature $F_{I_{r,b}}(x)$ by the formula

$$F_{I_{r,b}}(x) = \frac{r^2(d\bar{x} \wedge dx)}{(1 + r^2|x - b|^2)^2}.$$

We note that as r tends to zero, $F_{I_{r,b}}$ gets concentrated around the centre b , the distribution of the curvature around b being determined by the scale $r \in \mathbf{R}^+$. The full group $\text{Conf}(\mathbf{R}^4)$ of orientation-preserving conformal transformations of \mathbf{R}^4 acts on $I(0,1)$ to produce instantons of chosen centre and scale. On the other hand, the instanton $T_{r,b}^*I$ is invariant under the action of linear transformations $a_{u,v} : \mathbf{H} \rightarrow \mathbf{H}$ defined by $a_{u,v}(x) = uxv^{-1}$, which are proved to generate the group $SO(4)$.

The more difficult result that every 1-instanton is gauge equivalent to one of the family $I_{r,b}$ is proved in [1]. Hence, the moduli space $M_1(S^4)$ of 1-instantons is the open 5-ball $B^5 = \text{Conf}(\mathbf{R}^4)/SO(4)$, the centre of B^5 being occupied by the standard instanton I and the point along the radius to $b \in S^4$ at distance r from the centre being the instanton $(T_{1-r,b})^*I$ [15].^{†12}

The construction of multi-instantons: The *ADHM* construction^{†13} gives a correspondence between solutions of the *ASD* Yang-Mills equations over S^4 and certain systems of finite dimensional algebraic data indexed by an integer k .

We already used the *ADHM* method to describe explicitly instantons of charge $k = 1$: we chose a suitable quaternionic function in order to define the basic 1-instanton $I(0,1)$ and then used the transitive action of dilations and translations of \mathbf{R}^4 on the moduli space to construct a 2-parameter family $\{I(r,b)\}$ of 1-instantons.

We now survey the construction of instantons for larger values of k . A classic reference on the subject is Atiyah's monograph [1]. Let \mathbf{H}^k be the space of column quaternionic k -vectors. We define an $SU(2)$ -connection on \mathbf{H}^k by the formula

$$A(u) = \text{Im}\left\{\frac{u^* du}{1 + |u|^2}\right\} \quad (1.2)$$

where u^* denotes the transposed conjugate of $u \in \mathbf{H}^k$. We note that the formula (1.2) restricts to formula (1.1) on any 1-dimensional \mathbf{H} -subspace of \mathbf{H}^k . We choose a

^{12†} Note that we identify $T_{r,b}I$ with $T_{1/r,b^*}I$, b^* being the antipodal of b .

^{13†} The acronym is derived from the initials of those who introduced the construction, namely, Atiyah, Drinfeld, Hitchin and Manin.

function $u : \mathbf{H} \longrightarrow \mathbf{H}^k$ defined by $u(x) = r(B - x)^{-1}$, where B is a symmetric $k \times k$ matrix of quaternions and $r = (r_1, \dots, r_k)$, $r_i \in \mathbf{H}$.

For $k = 1$ the parameters were arbitrary, except that r had to be invertible. However, in the general case the parameters r and B are required to satisfy the following algebraic constraints.

1. The ADHM equations: The $k \times k$ matrix $B^*B + r^*r$ has real entries, i.e. the coefficients of i, j, k in $B^*B + r^*r$ all vanish.

2. Non-degeneracy condition: For each $x \in \mathbf{H}$, the equations $(B - x)\xi = 0$ and $r\xi = 0$, $\xi \in \mathbf{H}^k$ imply that $\xi = 0$, i.e. the $(k+1) \times k$ matrix $\begin{pmatrix} r \\ B - x \end{pmatrix}$ is of maximal rank k , for all $x \in \mathbf{H}$.

The first constraint ensures that the connection $A_{r,B}$ defined by substituting $u(x) = r(B - x)^{-1}$ onto (1.2) is ASD, whilst the latter ensures that $A_{r,B}$ is non-degenerate.^{†14} The proof that the ADHM method produces all Yang-Mills fields over S^4 is a hard task and requires a wide range of techniques which are explained analytically in [1]. For clarity, we state the main theorem.

Theorem 1.3.1 *Every ASD $SU(2)$ k -instanton over S^4 , arises from parameters (r, B) that satisfy (1) $B^*B + r^*r$ is a real $k \times k$ matrix, (2) for every $x \in \mathbf{H}$, the equations $(B - x)\xi = 0$ and $r\xi = 0$, $\xi \in \mathbf{H}^k$, imply $\xi = 0$. The connections defined by (r, B) and (r', B') are gauge equivalent if and only if $r' = qrT$, $B' = T^{-1}BT$, where $q \in Sp(1)$, $T \in O(k)$.*

Hence, the ADHM construction identifies the moduli space of equivalence classes of matrix data (with respect to the action of $O(k) \times SU(2)$) which satisfy both the constraints set above with the space $M_k(S^4)$ of gauge equivalence classes of ASD $SU(2)$ -connections of charge k .

1.3.1 Centred instantons

We recall Taubes' definition of centred instantons on S^4 with its standard round metric g_1 [41]. Let A be a k -instanton over S^4 and $x = \phi_n^{-1} : S^4 \setminus \{s\} \longrightarrow \mathbf{R}^4$ be the stereographic projection from the south pole. We pull back the instanton A through x to obtain a

^{14†} In particular, the points x for which $B - x$ is singular, give singularities that can be removed by a gauge transformation.

δ -ASD connection A over \mathbf{R}^4 (with its standard metric δ).

Definition 1.3.2 Let $A \in M_k(S^4)$ be as above. The *mass centre* q and the *scale* λ of A are defined by

$$q[A] = \frac{1}{8\pi^2 k} \int_{\mathbf{R}^4} x |F_A|^2 d^4x, \quad \lambda^2[A] = \frac{1}{8\pi^2 k} \int_{\mathbf{R}^4} |x - q| |F_A|^2 d^4x \quad (1.3)$$

We say that A is *centred* if it is non-flat and obeys the equalities $q[A] = 0$ and $\lambda[A] = 1$. For the flat connection C we make the convention $q[C] \equiv 0$ and $\lambda[C] \equiv 0$.

Remark 1.3.3 (a) Equations (1.3) yield the inequality $\int_{|x-q| \geq R\lambda} |F_A|^2 d^4x \leq \frac{8\pi^2 k}{R^2}$, also known as *Tchebychev inequality*. The moral is that an increase to the radius of the ball $B_q(\lambda)$ by R results to the increase of the curvature of A over $B_q(R\lambda)$ by a factor greater than or equal to $8\pi^2 k/R^2$.

(b) Let M_k^0 be the moduli space of centred instantons over S^4 . Note that M_1^0 consists of a single point, the standard 1-instanton $I(0, 1)$. For any $k > 0$, M_k^0 is a smooth submanifold of the moduli space M_k . Moreover, M_k is diffeomorphic to the product $M_k^0 \times \mathbf{R}^4 \times (0, \infty)$.^{†15}

1.4 Compactified moduli spaces

The moduli space \mathcal{M}_k of ASD connections over a simply-connected, Riemannian 4-manifold (X_0, g_0) is usually non-compact. However, one can follow K. Uhlenbeck's scheme of adding 'ideal' boundary points in order to compactify \mathcal{M}_k . This section outlines the definition of the Uhlenbeck compactification and also describes how neighbourhoods of points at infinity look like.

Definition 1.4.1 An *ideal g_0 -ASD connection* of charge k , is a pair $([A], x_1, \dots, x_l)$, where $[A] \in \mathcal{M}_{k-l}$ and (x_1, \dots, x_l) is a (possibly empty) l -tuple of (not necessarily distinct) points in X .

Definition 1.4.2 A sequence A_a in \mathcal{M}_k *converges weakly* to an ideal ASD connection $([A], x_1, \dots, x_l)$, if the following hold.

(a). There are C^∞ bundle maps $\zeta_a : P_{k-l}|_{X_0 \setminus \{x_1, \dots, x_l\}} \longrightarrow P_k|_{X_0 \setminus \{x_1, \dots, x_l\}}$ such that

¹⁵† For the proof, we refer to §3 of [41] or §3.2 of [14].

$\zeta_a^*(A_a)$ converges (in C^∞) on compact subsets to A .

(b). The action densities $\text{Tr}(F_{A_a}^2)$ converge as measures to $\text{Tr}(F_A^2) + \sum_{i=1}^l \delta_{x_i}$, where δ are delta functions at x_i .

Through the natural extension of Definition 1.4.1 to sequences of ideal connections, the set $\mathcal{IM}_k = \coprod_{q=0}^k \mathcal{M}_{k-q} \times Sp^q(X)$ of ideal g_0 -ASD connections of charge k is given a second-countable, Hausdorff and metrizable topology. Let $\bar{\mathcal{M}}_k$ be the closure of \mathcal{M}_k into \mathcal{IM}_k with respect to this topology. It is due to Uhlenbeck's *weak compactness theorem* that the space $\bar{\mathcal{M}}_k$ is compact, since every infinite sequence in \mathcal{M}_k has a weakly convergent subsequence with limit point in $\bar{\mathcal{M}}_k$. The details of the proof take up Section 4.4 of [8] and §8 of [15].

Let us now analyze the structure of the 'ends' of $\bar{\mathcal{M}}_k$ by using the gluing construction of Section 1.2. We pick a point $([A_0], x_1, \dots, x_l) \in \bar{\mathcal{M}}_k \cap (\mathcal{M}_{k-l} \times Sp^l(X))$ such that the tuple (x_1, \dots, x_l) lies away from the diagonal of $Sp^l(X)$, namely each point x_i has multiplicity 1. One can then prove that every point $[A]$ that is 'close' to $([A_0], x_1, \dots, x_l)$ with respect to Uhlenbeck's 'weak' topology lies in a *gluing neighbourhood*. More precisely, let $[A_a]$ be a sequence in \mathcal{M}_k which converges weakly to $([A_0], x_1, \dots, x_l)$. In the way described in §4.2 of [14] and §8.2.1 of [8], the sequence $[A_a]$ produces sequences of local mass centres x_{ia} and scales λ_{ia} such that $x_{ia} \rightarrow x_i$ and $\lambda_{ia} \rightarrow 0$, $1 \leq i \leq l$.

A λ_{ia} -dilation of g_0 around x_{ia} produces a sequence $\{g_a\}$ of conformally equivalent metrics on $X = X_0 \#_{i=1}^l X_i$, where $X_i = S^4$. This *conformal blow-up* process produces a sequence of g_a -ASD connections, $[\check{A}_a]$ say, which converges *strongly*¹⁶ to a limit (A_0, I_1, \dots, I_l) over the join $X_0 \vee_{i=1}^l X_i$, where the standard 1-instantons I_j are centred at the north pole n and have scale 1.

In this way, we have no 'singular' points in the sense that there is no curvature loss over the necks. To obtain an open neighbourhood of $([A_0], x_1, \dots, x_l)$ in $\bar{\mathcal{M}}_k$ one simply glues up the limit (I_1, \dots, I_l) . A summary on the description of the structure of the ends of the compactified moduli space can be found in §8.2.2 of [8].

¹⁶† Strong convergence requires C^∞ convergence over compact sets away from the necks and also $c_2(A_0) + \sum_{i=1}^l c_2(I_i) = k$.

Remark 1.4.3 An ambiguity appears in the preceding discussion when one attempts to describe neighbourhoods of boundary points $([A_0], x_1, \dots, x_l)$ with the l -tuple (x_1, \dots, x_l) lying on the diagonal of the symmetric product $Sp^l(X_0)$. In this case, as we shall see in Section 1.6, the limiting behaviour of the sequence $[A_a]$ is more complicated due to a possible loss of curvature over the gluing necks. More precisely, the corresponding sequence $[\check{A}_a]$ converges now weakly to an ideal limit into the ideal moduli space $\mathcal{IM}_k(S^4)$ of k -instantons over S^4 . An *iterated* conformal blow-up procedure is then required to ensure that after a finite number of steps, the sequence $[\check{A}_a]$ converges strongly to a limit $\{A_I\}$ over the join $X_0 \vee_I X_I$.

1.5 The gluing setting of the thesis

We describe the choices of gluing data required for our purposes and discuss the importance of introducing a *universal gluing map* which incorporates all possible variations of the existing gluing constructions.

1.5.1 About the first step

Let us deal with the first level of gluing, namely to glue a single copy of the standard 1-instanton I over S^4 onto a background connection of a reference 4-manifold X_0 . So, let \mathcal{A}_{k-1} be the space of *ASD* connections on an $SU(2)$ -bundle P_{k-1} over a simply-connected, Riemannian 4-manifold (X_0, g_0) and \mathcal{G}_{k-1} be the group of gauge transformations of P_{k-1} . Let \mathcal{M}_k be the *ASD* moduli space of charge k .

The structure group $SU(2)$ acts on P_{k-1} in the obvious way and on the quaternions \mathbf{H} by multiplication on the left. With respect to these actions we form the associated quaternionic bundle $\pi : E_{k-1} \rightarrow X_0$ defined by $E_{k-1} = P_{k-1} \times_{SU(2)} \mathbf{H}$ and let $|\cdot|$ denote the induced inner product on E_{k-1} . We aim to construct an embedding

$$T_1 : \mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1} \longrightarrow \mathcal{IM}_{k,1} = M_k \coprod M_{k-1} \times X_0 .$$

We pick $([A_0], [p]) \in \mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1}$ and choose $X_1 = S^4$ with its standard round metric g_1 . We also choose $x_1 = \pi(p) \in X_0$ and $x_2 = \infty = n \in S^4$. Let $A_1 = [I]$ be the equivalence class of the basic 1-instanton I on S^4 - see Section 1.3.

Since the construction is non-canonical, we need to specify the other parameters which are involved. More precisely, we define the *neck-size* parameter λ_p by

$$\lambda_p = \begin{cases} |p|, & \text{if } |p| < 1 \\ \frac{1}{|p|}, & \text{if } |p| \geq 1, \quad p \in E_{k-1}. \end{cases}$$

With base points and scales in hand, one defines a conformal diffeomorphism f_1 that identifies the associated annuli centred at $x_i, i = 1, 2$. The connected sum $X = X_0 \cup_{f_1} S^4$ is then formed in the familiar way - see Section 1.3. Let us also define a gluing parameter $\varrho : (E_{k-1})|_{x_1} \rightarrow P_1|_\infty$ by setting $\varrho(q) = q/|q|$. The application of the gluing construction for single connected sums, described in Section 1.2, yields an approximately *ASD* connection A denoted by $A = T_1(\varrho, \pi(p), \lambda_p) \equiv A_0 \#_{p, \lambda_p} I$. Let us recall that the gauge equivalence class $[A]$ consists of all elements $A(\rho)$ with $\rho \in Gl = Hom_{SU(2)}((E_{k-1})|_x, P_1|_\infty)$ lying on the orbit of ϱ with respect to the action of the product $\Gamma_{A_0} \times \Gamma_I$ of the isotropy groups on Gl .

Let E_{k-1}^* denote the quaternionic bundle E_{k-1} with its zero section removed and consider the set E_{k-1}^0 defined by $E_{k-1}^0 = \{p \in E_{k-1} : \lambda_p = 0\}$. We note that if $p \in E_{k-1}^*$, the glued connection $A = A_0 \#_{p, \lambda_p} I$ lies in \mathcal{M}_k . On the other hand, if p lies on the zero section of E_{k-1} , by construction the map T_1 becomes the identity $Id : \mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1}^0 \rightarrow \mathcal{M}_{k-1} \times X_0$. In this sense, T_1 is roughly viewed as

$$T_1 : \mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1}^* \coprod \mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1}^0 \rightarrow \mathcal{M}_k \coprod \mathcal{M}_{k-1} \times X_0.$$

To see that T_1 is an embedding, one recalls the discussion of Section 1.4: a neighbourhood of an ideal connection on the boundary of $\bar{\mathcal{M}}_k$ can be constructed by applying the gluing procedure on appropriate data. Each point in the gluing neighbourhood is determined by a centre and a scale which in our case are coupled to $\pi(p)$, $p \in E_{k-1}$ and $\lambda_p \in \mathbb{R}^+$.

1.5.2 Universal multiple-gluing maps

Let P_{k-l} be a principal $SU(2)$ -bundle of 2nd Chern class $k-l$ over a simply-connected 4-manifold X_0 . With respect to the action of $SU(2)$ on the quaternions \mathbf{H} , we consider the associated quaternionic bundle $\pi : E_{k-l} \rightarrow X_0$ by setting $E_{k-l} = P_{k-l} \times_{SU(2)} \mathbf{H}$. Let $Sp^l(E_{k-l})$ be the l -th symmetric product of E_{k-l} , defined by

$$Sp^l(E_{k-l}) = \frac{E_{k-l} \times E_{k-l} \times \dots \times E_{k-l}}{S_l}.$$

The quotient space $Sp^l(E_{k-l})$ is a bundle over $Sp^l(X)$ with fiber $Sp^l(\mathbf{H})$. Let \mathcal{A}_{k-l} be the space of ASD connections on P_{k-l} and \mathcal{G}_{k-l} be the group of gauge transformations of P_{k-l} . With respect to the natural action of \mathcal{G}_{k-l} on \mathcal{A}_{k-l} and E_{k-l} we form the quotient space $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})$.

Let $\mathcal{IM}_{k,l}$ denote the union $\mathcal{IM}_{k,l} = \coprod_{q=0}^l \mathcal{M}_{k-q} \times Sp^q(X)$ which as explained in Section 1.4 can be given a second countable and Hausdorff topology through the notion of weak convergence of sequences of connections to ideal points. The main aim of the chapter is to prove that if X_0 and E_{k-l} are as above, then there is a gluing map $\mathcal{F}_l : \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}) \longrightarrow \mathcal{IM}_{k,l}$ such that (a) \mathcal{F}_l is defined for all $1 \leq l \leq k$, (b) \mathcal{F}_l is a diffeomorphism onto its image.

More precisely, let us pick a point $([A_0], [p_1, \dots, p_l]) \in \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})$. As long as the points (p_1, \dots, p_l) lie away from the diagonal of $Sp^l(E_{k-l})$, we explicitly define a gluing map in the following way. We choose several copies of the standard 1-instanton on S^4 , namely $I_1 = I_2 = \dots = I_l = I(0,1)$. Following the gluing procedure of Section 1.2, we graft I_1, \dots, I_l onto A_0 at the base points $x_1 = \pi(p_1)$, $x_2 = \pi(p_2)$, \dots , $x_l = \pi(p_l)$ and $y = \infty = n \in S^4$, using the scales $\lambda_{p_1}, \dots, \lambda_{p_l}$ defined in §1.5.1. We recall that the main feature of the above grafting is that the points x_1, \dots, x_l must have multiplicity 1. Consequently, the procedure defines a map $T_l : \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} D^l(E_{k-l}) \longrightarrow \mathcal{M}_k$, where $D^l(E_{k-l}) \subset Sp^l(E_{k-l})$ is defined by $D^l(E_{k-l}) = \{[p_1, p_2, \dots, p_l] : p_j, p_i \text{ lie on distinct fibers, } i \neq j, 1 \leq i, j \leq l\}$.

One would naturally pose the problem of defining a gluing map, \mathcal{F}_l say, in the case that some of the points (p_1, p_2, \dots, p_l) lie on the same fiber of E_{k-l} , i.e. when the base points on X_0 appear with multiplicities greater than or equal to 2. The obstacle which appears then concerns certain ‘energy estimates’ which would guarantee the anti-self-duality of the resulting glued connection and also the description of neighbourhoods of ideal boundary points of the moduli space \mathcal{M}_k which lie on the diagonal.

We shall show that such a gluing map $\mathcal{F}_l : \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}) \longrightarrow \mathcal{M}_k$ indeed exists and it is *universal* in the sense that its restriction to the parameter space $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} D^l(E_{k-l})$ agrees with T_l .

1.6 Bubble-tree method and gluing maps

Let \mathcal{M}_k be the moduli space of g_0 -ASD connections over a 4-manifold X_0 . This section analyzes the behaviour of sequences of \mathcal{M}_k which converge weakly to an ideal point (A_0, x_1, \dots, x_l) , where the tuple (x_1, \dots, x_l) lies on the diagonal of $Sp^l(X_0)$. The results lead to the definition of a universal multiple-gluing map as mentioned in §1.5.2. The material presented here has been taken from §4.2 of [14].^{†17}

A reminder of the problem and a synopsis of the method: Let $[A_a]$ be a sequence in \mathcal{M}_k that converges weakly to the ideal limit $([A_0], x_1, \dots, x_l) \in \mathcal{M}_{k-l} \times Sp^l(X_0)$. In §1.6.1, we show that $[A_a]$ produces sequences of local mass centres x_{ia} and scales λ_{ia} such that $x_{ia} \rightarrow x_i$ and $\lambda_{ia} \rightarrow 0$, $1 \leq i \leq l$. We use λ_{ia} to dilate the Riemannian metric g_0 around x_{ia} producing a sequence $\{g_a\}$ of conformally equivalent metrics on the connected sum $X = X_0 \#_{i=1}^l X_i$, where $X_i = S^4$. This conformal blow-ups produce a sequence of g_a -ASD connections $[\check{A}_a]$ over X , which provided that the points (x_1, \dots, x_l) are distinct converge strongly to a limit (A_0, I_1, \dots, I_l) over the join $X_0 \vee_{i=1}^l X_i$, I_j being a copy of the standard 1-instanton over $X_i = S^4$.

On the other hand, if the tuple (x_1, \dots, x_l) lies on the diagonal of $Sp^l(X)$, then the limiting behaviour of $[A_a]$ is quite different: the sequence $[\check{A}_a]$ converges now over compact subsets of $X_0 \setminus \{x_1, \dots, x_l\}$ to a g_0 -ASD connection A_0 , although converges only weakly to (A_i, Z_i) over the join $\vee_{i=1}^l X_i$. As we explain in due course, provided that the sets $Z_i = \{x_{i1}, \dots, x_{il_i}\}$, $1 \leq i \leq l$, are empty, one can glue up the limits $\{A_i\}_{i=1}^l$ to produce an open set of $\mathcal{M}_{X_0, k}$.

However, if $Z_i \neq \emptyset$ one needs to iterate the conformal blow-ups in order to ensure that after a finite number of steps the sequence $[\check{A}_a]$ converges strongly to a limit $\{A_I\}_{I \in \mathcal{I}}$ over the join $X_0 \vee_{I > 0} X_I$.

In this way, we guarantee that all singular points are finally blown away and that no curvature loss occurs over the gluing necks. We then apply the known techniques to glue the bubble-tree limits $(A_I)_{I \in \mathcal{I}}$ onto A_0 . We therefore obtain an ASD connection over $X = \#_{I \in \mathcal{I}} X_I$ and consequently we construct open subsets of the moduli space $\mathcal{M}_{X, k}$

^{†17} The initial goal of [14] is the proof of that the moduli space \mathcal{M}_{k, X_0} of irreducible g_0 -ASD connections of a Riemannian 4-manifold X_0 has finite volume and diameter with respect to the L^2 metric g defined by g_0 - see §1(c).

by varying the limit data. To describe open subsets of $\mathcal{M}_{X_0, k}$, we pull back through the family of conformal diffeomorphisms f_I determined by the procedure.

1.6.1 Conformal blow-ups

Let $[A_a]$ be a sequence in \mathcal{M}_k converging weakly to the ideal limit $([A_0], x_1, \dots, x_l)$, where the set $Z_0 = \{x_1, \dots, x_l\}$ lies on the diagonal of $Sp^l(X)$. Let us assume that the point x_i has multiplicity $k_i \geq 1$. Then, we note that

$$\lim_{r \rightarrow \infty} \lim_{a \rightarrow \infty} \int_{B_{x_i}(r)} |F_{A_a}|^2 dV_{g_0} = 8\pi^2 k_i .$$

We define sequences of local mass centres and scales associated to $[A_a]$ and use them to define a sequence of conformal diffeomorphisms which ‘resolve’ the singular points x_i . More precisely, let r_0 be the injectivity radius of X_0 , $d_0 \leq \min_{i \neq j} \text{dist}_{g_0}(x_i, x_j)$ and $s_0 < \frac{1}{4} \min\{1, r_0, d_0\}$. Choose an orthonormal frame $v_i \in Fr X_0$ and let $q = \phi_{x_i}^{-1}$ be the associated Gaussian coordinate system. For each i , we define mass centres and scales associated to $[A_a]$ restricted to the fixed balls $B_{x_i}(s_0)$ as follows.^{†18}

Definition 1.6.1 Let $[A_a], x_i, s_0, q$ be as above. For each i , a sequence of *mass centres* $\{x_{ia}\}_{a=1}^\infty \in B_{x_i}(s_0)$ is defined by $x_{ia} = \phi_{x_i}(q_{ia})$, where

$$q_{ia} = \text{Centre}(A_a|_{B_{x_i}(s_0)}) \equiv \frac{1}{8\pi^2 k_i} \int_{B_{x_i}(s_0)} q(|F_{A_a}|^2 - |F_{A_0}|^2) dV_{g_0} \quad (1.4)$$

For each i , a sequence of *scales* $\{\lambda_{ia}\}_{a=1}^\infty \in (0, \infty)$ is defined by

$$\lambda_{ia}^2 = \text{Scale}^2(A_a|_{B_{x_i}(s_0)}) \equiv \frac{1}{8\pi^2 k_i} \int_{B_{x_i}(s_0)} |q - q_{ia}|(|F_{A_a}|^2 - |F_{A_0}|^2) dV_{g_0} \quad (1.5)$$

Remark 1.6.2 We note that the above definition leads to the Tchebychev inequality

$$\int_{B_{x_i}(s_0) \setminus B_{x_{ia}}(R\lambda_{ia})} (|F_{A_a}|^2 - |F_{A_0}|^2) dV_{g_0} \leq \frac{8\pi^2 k_i}{R^2} . \quad (1.6)$$

Hence, for $R \gg 1$ and a large, the balls $B_{x_{ia}}(R\lambda_{ia})$ contain most of the $8\pi^2 k_i$ A_a -energy bubbling off at x_{ia} .

^{18†} Note that $\lim_{a \rightarrow \infty} \int_{B_{x_i}(s_0)} (|F_{A_a}|^2 - |F_{A_0}|^2) dV_{g_0} = 8\pi^2 k_i$.

One then deduces that $\{x_{ia}\} \rightarrow x_i$ and $\{\lambda_{ia}\} \rightarrow 0$. Let $\{v_{ia}\} \in FrX_0$ be a sequence that converges to v_i . Following the scenario of Section 1.2, we define diffeomorphisms $f_{x_{ia}} = \phi_{in} \circ d_{\lambda_{ia}} \circ \phi_{x_{ia}}^{-1}$, where $\phi_{x_{ia}}^{-1}$ are the corresponding Gaussian charts.^{†19} Let X'_{ia} be open subsets of X_{ia} as described in Section 1.2 and \bar{g}_{ia} be metrics as described in Remark 1.2.3.

We recall that \bar{g}_{ia} is quasi-conformally equivalent to the round metric of S^4 with respect to which the identification maps $f_{x_{ia}}$ are conformal. Let $A_{ia} = f_{x_{ia}}^* A_a$ be the induced sequence on $P_{ia} = f_{x_{ia}}^* P$. Let $\{g_a\}$ be a sequence of smooth metrics on the connected sum $X = X'_{0a} \#_{i=1}^l X'_{ia}$ and also \check{A}_a be the induced g_a -ASD sequence over X . It is not hard to see that the L^2 -Sobolev norm of the curvature $|F(A_{ia})|$ over X'_{ia} satisfies the equation

$$8\pi^2(k_i + 1/2) \leq \int_{X_{ia}} |F_{A_{ia}}|^2 dV_{\bar{g}_{ia}} = \int_{B_{x_{ia}}(N\sqrt{\lambda_{ia}})} |F_{A_a}|^2 dV_{g_0} \leq 8\pi^2(k_i - 1/2) \quad (1.7)$$

With the above data given, Proposition 4.3 of §4.1 of [14] asserts that the sequence $\{A_{ia}\}$ has a subsequence, relabelled as $\{A_{ia}\}$, which converges weakly to an ideal g_i -ASD connection (A_i, Z_i) over X_i , $1 \leq i \leq l$, where $Z_i = \{x_{ij}\}_{j=1}^{l_i}$.^{†20} Equation (1.7) implies that (A_i, Z_i) has mass $8\pi^2 k_i$ where $k_i = \sum_{j=1}^{l_i} k_{ij}$, each point x_{ij} having multiplicity k_{ij} . We also note that A_i is a g_i -ASD connection on a bundle $P_i \rightarrow X_i$ of 2nd Chern class k_{i0} .

Higher level blow-ups: Let us assume that the singular sets Z_i appearing on the ideal limits (A_i, Z_i) described above are non-empty. As we shall shortly see, in this case the conformal blow-up process needs to be *iterated* such that after applying the process at most k times, we obtain a sequence of g_a -ASD connections \check{A}_a over X which is strongly convergent.

More precisely, the corresponding sequence $\{A_{ia}\}$ converges weakly over the open subset X'_{ia} near a point x_{ij} of multiplicity k_{ij} in the set $Z_i \subset X_i$. We apply the *second level conformal blow-ups* by working as above except in minor details: we define sequences of mass centres $x_{ija} = \phi_{x_{ij}}(q_{ija})$ converging to x_{ij} and scales λ_{ija} converging to 0, by using the almost round metrics \bar{g}_{ia} and a coordinate chart $\phi_{x_{ij}} = \phi_{in} \circ \tau_{q_{ij}}^{-1}$,

¹⁹† Recall that $c_{\lambda_{ia}}(x) = x/\lambda_{ia}$, $x \in \mathbb{R}^4$.

²⁰† Note that Equation (1.6) ensures that $Z_i \subset X_i \setminus x_{is}$.

where $\phi_{in}(q_{ij}) = x_{ij}$ and $\tau_{q_{ij}}$ denotes the translation by q_{ij} . The blow-up conformal diffeomorphisms are then defined using the systems $\phi_{x_{ij,a}} = \phi_{in} \circ \tau_{q_{ij}}^{-1}$ and setting $f_{x_{ij,a}} = \phi_{ijn} \circ c_{\lambda_{ij,a}} \circ \phi_{x_{ij,a}}^{-1}$. For all higher level blow-ups, we carry out the process described in §1.6.1, until we produce a sequence $\{A_{i,a}\}$ that converges weakly to an ideal limit (A_i, Z_i) , where Z_i is empty.

Centred limits: To serve our later needs, we present a method of choosing the conformal blow-ups of (X_0, g_0) in such a way that the g_i -ASD limit (A_i, Z_i) is *centred* in the sense of §1.3.1. The main result can be found in [14] and is displayed below for completeness.

Theorem 1.6.3 *Let $[A_a]$ be a sequence of g_0 -ASD connections that converges weakly to $([A_0], x_1, \dots, x_l)$, where $Z_0 = \{x_1, \dots, x_l\}$ lies on the diagonal of $Sp^l(X)$. We choose s_0 as in Definition 1.6.1. For each point x_i , the sequence $[A_a]$ determines (a) a sequence $\{y_{i,a}\}$ of points that converges to x_i , (b) a sequence $\{r_{i,a}\}$ of scales that converges to 0, (c) a sequence $\{u_{i,a}\} \in FrX_0|_{y_{i,a}}$ of frames that converges to $v_i \in FrX_0|_{x_i}$, with the following significance: let $f_{y_{i,a}}$ be the corresponding sequence of conformal blow-ups and let $A_{y_{i,a}}$ be the induced sequence of $\bar{g}_{y_{i,a}}$ -ASD connections with weak g_i -ASD limit (A_i, Z_i) over the 4-sphere X_i .*

The limit (A_i, Z_i) has then the following properties: (a). if A_i is not the product connection C , then A_i is centred, (b). if $A_i = C$, then the corresponding curvature measure of (A_i, Z_i) is centred in the sense of §4.2 of [14].

For the remainder of the chapter, we require the conformal blow-ups to be chosen as in Theorem 1.6.3.

Remark 1.6.4 Let $[A_a]$ be a sequence of g_0 -ASD connections over X_0 that converges weakly to $([A_0], Z_0)$, where $Z_0 = \{x_1, \dots, x_l\} \subset Sp^l(X)$. Let $A_{i,a}$ be the induced sequence of $\bar{g}_{x_{i,a}}$ -ASD connections over $X'_{i,a}$ with weak g_i -ASD limit (A_i, Z_i) over the 4-sphere X_i . One can prove that given $\epsilon > 0$ and $N > 4$, there exists an index a_0 such that for any $a > a_0$ it holds that

$$\|F(A_a)\|_{L^2(\Omega_{i,a}, g_0)} < \epsilon \quad \text{and} \quad \left| \|F(A_a)\|_{L^2(B'_{i,a}, g_0)} - 8\pi^2 k_i \right| < \epsilon,$$

where $\Omega_{i,a} = \Omega(x_{i,a}, N^{-1}\sqrt{\lambda_{i,a}}, N\sqrt{\lambda_{i,a}})$ and $B'_{i,a} = B_{x_{i,a}}(N\sqrt{\lambda_{i,a}})$. This implies that in

the limit no curvature loss occurs over the gluing necks Ω_i .^{†21}

Bubble-tree ideal connections and convergence: To describe the methodology of [14], we adapt the notion of ideal *ASD* connections and convergence of Section 1.4 to the case of bubble-trees. Throughout, we recall the terminology introduced in Section 1.2.

Definition 1.6.5 A *bubble-tree ideal g_0 -ASD* connection of charge k over the 4-manifold X_0 , is determined by the following data.

- (1). A finite tree \mathcal{I} consisting of m vertices, each vertex being indexed by an integer $k_I \geq 0$, such that (a) $\sum_{I \in \mathcal{I}} k_I = k$, (b) if the vertex I is terminal, then k_I is greater than 0, (c) there are at most k terminal vertices, excluding the base vertex 0.
- (2). A $(2m-1)$ -tuple $(A_I, x_I)_{I \in \mathcal{I}}$ such that (a) if $I = 0$, then A_0 is a g_0 -ASD connection of charge $k_0 \geq 0$, over the 4-manifold X_0 , (b) if $I > 0$, then A_I is either the trivial product connection C or a *centred g_I -ASD* connection of charge k_I , over the 4-sphere X_I with its standard round metric g_I , (c) if $I = 0$, then $x_I \in X_0$ and if $I > 0$, then $x_I \in X_{I-} \setminus \{x_{I_s}\}$.
- (3). If $I > 0$ and $A_I = C$, then there at least 2 outgoing edges emerging from I .

A bubble-tree ideal connection $(A_I, x_I)_{I \in \mathcal{I}}$ can be roughly viewed as a ‘connection’ over the join $\bigvee_{I \in \mathcal{I}} X_I$, each sphere X_I being attached to X_{I-} by identifying the south pole x_{I_s} with $x_I \in X_{I-}$.

Definition 1.6.6 Let Z_I be the set of the ‘attachement points’ x_I mentioned above. Let $\{A_a\}$ be a sequence of g_a -ASD connections of charge k , over the multiple connected sum $X = \#_{I \in \mathcal{I}} X_I$.

- (1). Let $Y \in Sp^k(X)$ be a symmetric k -tuple in $\bigcup_{I \in \mathcal{I}} X_I \setminus (Z_I \cup \{x_{I_s}\})$. Then, $\{A_a\}$ *converges weakly* to $((A_I, x_I)_{I \in \mathcal{I}}, Y)$, if $[A_a]$ converges in C^∞ to $(A_I, x_I)_{I \in \mathcal{I}}$ over compact sets of $\bigcup_{I \in \mathcal{I}} X_I \setminus (Z_I \cup \{x_{I_s}\} \cup Y)$ and if the curvature densities converge as measures over $\bigcup_{I \in \mathcal{I}} X_I \setminus (Z_I \cup \{x_{I_s}\})$, i.e. $|F(A_a)|_{g_a}^2 \longrightarrow \sum_{I \in \mathcal{I}} |F(A_I)|_{g_I}^2 + 8\pi^2 \delta_Y$, where δ_Y denote delta functions at elements of Y .

- (2). The sequence $\{A_a\}$ *converges strongly* to $(A_I, x_I)_{I \in \mathcal{I}}$, if it converges weakly to $((A_I, x_I)_{I \in \mathcal{I}})$ and also $\sum_{I \in \mathcal{I}} k_I = k$, where k_I is the charge of A_I .

^{21†} We refer to §4.2 of [14] and Section 7.3 of [8].

Definition 1.6.7 Let \mathcal{BM}_k be the space of bubble-tree ideal g_0 -ASD connections over X_0 of total charge k . Let $\{A_a\}$ be a sequence of g_0 -ASD connections of charge k , over X_0 . We say that $\{A_a\}$ converges to a bubble-tree g_0 -ASD connection $(A_I, x_I)_{I \in \mathcal{I}} \in \mathcal{BM}_k$, if there exist sequences of conformal blow-ups $\{f_{I_a}\}_{I \in \mathcal{I}}$ with the following significance: if \tilde{A}_a are the induced g_a -ASD connections over the connected sum X , then we require that the sequence of metrics $\{g_a\}$ converge in C^∞ on compact subsets of $X_I \setminus (Z_I \cup \{x_{I_s}\})$ to the metric g_I , $I \geq 0$, and that the sequence \tilde{A}_a converges strongly to the ideal g_0 -ASD connection $(A_I, x_I)_{I \in \mathcal{I}}$.

Remark 1.6.8 Let $\tilde{\mathcal{M}}_{X_0, k}$ be the closure of $\mathcal{M}_{X_0, k}$ into \mathcal{BM}_k with respect to the convergence introduced in Definition 1.6.7. In [14], it is shown that the moduli space $\tilde{\mathcal{M}}_{X_0, k}$ is compact, since *any infinite sequence in $\mathcal{M}_{X_0, k}$ has a strongly convergent subsequence with limit point in $\tilde{\mathcal{M}}_{X_0, k}$* . Moreover, the compactification $\tilde{\mathcal{M}}_{X_0, k}$ maps to the Uhlenbeck compactification $\bar{\mathcal{M}}_{X_0, k}$ by sending a bubble-tree ideal connection $(A_I, x_I)_{I \in \mathcal{I}}$ to the corresponding Uhlenbeck ideal connection (A_0, x_1, \dots, x_l) . Corollary 4.16 of [14] states that the projection $b : \tilde{\mathcal{M}}_{X_0, k} \rightarrow \bar{\mathcal{M}}_{X_0, k}$ is continuous.

1.6.2 Construction of gluing maps

The kernel of this chapter is the construction of multiple-gluing maps:

$$\mathcal{F}' : \mathcal{D}_c \equiv \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}) \longrightarrow \mathcal{M}_{X, k}^*, \quad \mathcal{F} : \mathcal{D}_c \equiv \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}) \longrightarrow \mathcal{M}_{X_0, k}^*.$$

The subscript ‘ c ’ means to recall that the parameter space of gluing data contains g_I -ASD connections over $X_I = S^4$, $I > 0$, which are *centred* in the sense of Definition 1.3.2. In §1.6.1, we described a method of choosing the conformal diffeomorphisms f_I such that the ideal limits $(A_I)_{I \in \mathcal{I}}$ are centred.

The map \mathcal{F} will give rise to a universal gluing map $\mathcal{F}_l : \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}) \longrightarrow \mathcal{IM}_{k, l}$ which we construct in Section 1.7.

The framework: Let us consider $X = \#_{I \in \mathcal{I}} X_I$ with a C^∞ metric g which is conformally equivalent to the metric g_0 of X_0 . In Section 1.2, we used the gluing construction to define an ‘approximate’ gluing map $G : \mathcal{D} \rightarrow \mathcal{B}_{X, k}^*$ by setting $G(z) = [A'(z)]$. We then pulled back through the family $\{f_I\}$ of conformal diffeomorphisms to obtain almost ASD connections $\tilde{A}'(z)$ over X_0 .

We will now construct a family of *ASD* connections on X_0 , close to $\check{A}'(z)$ once the neck-width parameter determining the conformal structure of X is small.

The first task we deal with is the construction of a right inverse P to the operator $d_{A'}^{+,g} : \Omega^1(X, \text{ad}P) \longrightarrow \Omega^+(X, \text{ad}P)$. Let A_I be a g_I -*ASD* connection over X_I . By hypothesis, $H_{A_I}^2 = 0$, therefore the operators $d_{A_I}^{+,g_I}$ have right inverses P_I . We fix $2 \leq p < 4$, $4 \leq q < \infty$ such that $1/p - 1/q = 1/4$.

We set $P_I = (d_{A_I}^{+,g_I})^* G_{A_I}^{+,g_I}$, where $G_{A_I}^{+,g_I}$ is the Green's operator associated to the Laplacian $\Delta_{A_I}^{+,g_I} = d_{A_I}^{+,g_I} (d_{A_I}^{+,g_I})^*$. Lemma 5.1 of [14] gives bounds of the L^q, L^p, L_1^p norms of $P_I \xi$, namely there are constants $C_i(A_I, g_I, p)$, $i = 1, 2$, such that for all $\xi \in L^p \Omega^2(X_I, \text{ad}P_I)$, it holds that

$$\|P_I \xi\|_{L^q(X_I, A_I, g_I)} \leq C_1 \|P_I \xi\|_{L_1^q(X_I, g_I)} \leq C_2 \|\xi\|_{L^p(X_I, g_I)}.$$

To define a right parametrix Q for the operator $d_{A'}^{+,g}$, we need to patch together the right inverses P_I . To do so, we choose C^∞ cut-off functions β_I on each X_I as in Lemma 5.2 of [14]. Analytical estimates regarding the bounds of the error term R emerged by the cut-offs are presented in Lemma 5.3, 5.4 of [14] and also in §7.2 of [8]. The following proposition summarizes the construction of the right inverse P . For the remainder of the chapter, we choose $b_I = 4N\sqrt{\lambda_I}$.

Proposition 1.6.9 *There are constants N_0 and b_0 such that for any $N \geq N_0$, $\bar{b} \geq b_0$ and $z \in \mathcal{D}_c$, the operator $P = Q(1 + R)^{-1}$ is a right inverse to $d_{A'}^{+,g}$. Moreover, there are constants $C_1(g_0, p, \mathcal{D}_c), C_2(g_0, p, \mathcal{D}_c)$ such that for all $\xi \in L^p \Omega^2(X_I, \text{ad}P_I)$ it holds that*

$$\|P \xi\|_{L^p(X, g)} \leq C_1 \|P \xi\|_{L_1^p(X, g)} \leq C_2 \|\xi\|_{L^q(X, g)}.$$

Proposition 1.6.9 gives a desired uniform solution to the linearization of the *ASD* equation over X with respect to the scale parameters λ_I .

We now construct families of solutions to the full *ASD* equations over the connected sum X . More precisely, we seek solutions $A(z) = A'(z) + a(z)$ to the equation $F^{+,g}(A' + a) = 0$ or equivalently $d_{A'}^{+,g} a + (a \wedge a)^{+,g} = -F^{+,g}(A')$, $a \in \Omega^1(X, \text{su}(2))$. We look for solutions of the form $a = P \xi$, where $\xi \in \Omega^{+,g}(X, \text{su}(2))$. Therefore, the aforementioned *ASD* equation becomes

$$\xi + (P\xi \wedge P\xi)^{+,g} = -F^{+,g}(A') \quad (1.8)$$

Proposition 1.6.9 together with the application of Lemma 7.2.23 of [8]^{†22} to Equation (1.8) asserts that provided that the self-dual part $F^{+,g}(A')$ is sufficiently small, relative to constants depending only on A_I , there is a unique small solution ξ to Equation (1.8). On the other hand, we know from the energy estimates of Remarks 1.2.2, 1.2.4 that this condition can be achieved by making b small. The following theorem summarizes our discussion.

Theorem 1.6.10 *For any $z \in \mathcal{D}_c$, there exists an L_1^p g -ASD connection $A(z) = A'(z) + a(z)$ over X with $a(z) = P\xi(z)$. Moreover, there are constants $C_i = C_i(g_0, p, \mathcal{D}_c)$, $i = 1, 2, 3$, such that $\|a\|_{L^q(X,g)} \leq C_1 \|\xi\|_{L^p(X,g)} \leq C_2 \|F^{+,g}(A)\|_{L^p(X,g)} \leq C_3 / b^{4/p}$.*

Proof: We refer to §7.2.4 of [8] and §5.1 of [14].

We now pull back the g -ASD connections $A(z)$ through the conformal blow-ups f_I in order to obtain g_0 -ASD connections $\check{A}(z) = \check{A}'(z) + \check{a}(z)$, where \check{A} is defined by $\check{A} = f_0^* \dots f_I^* A$ over $f_0^{-1} \dots f_I^{-1}(X'_I)$ and similarly for \check{A}' and $\check{a}(z)$.

We note that $\check{A}(z) = \check{A}'(z) + \check{a}(z)$ solves the equation $d_{\check{A}'}^{+,g} \check{a} + (\check{a} \wedge \check{a})^{+,g} = -F^{+,g}(\check{A}')$. Moreover, one can copy standard arguments, to check that A and \check{A} are smooth connections of $\mathcal{M}_{X,k}$ and $\mathcal{M}_{X_0,k}$, respectively.^{†23} The following theorem summarizes the construction of the desired gluing maps.

Theorem 1.6.11 *Let A_I be g_I -ASD connections on $SU(2)$ bundles P_I over manifolds $X_I, I \in \mathcal{I}$. Let $X = \#_{I \in \mathcal{I}} X_I$ be the associated connected sum with a metric g which is determined by the choice of centres $\{x_I\}$, orthonormal frames $\{v_I\}$, scales $\{\lambda_I\}$ and fixed neck-size parameter N . Let P be the connected sum bundle over X of 2nd Chern class k and let $\bar{\lambda} = \max_{I \in \mathcal{I}} \lambda_I$. A space \mathcal{D}_c of gluing parameters can be chosen so that there is a C^∞ homeomorphism $\mathcal{G} : \mathcal{D}_c \rightarrow \mathcal{U} \subset \mathcal{M}_{X,k}^*$ onto an open set defined by $\mathcal{G}(z) = [A(z)]$, where $A(z) = A'(z) + a(z)$, $a(z) = P\xi(z)$ and $\xi(z)$ are as in Theorem 1.6.10. Moreover, for any $\nu > 0$ and $4 \leq q < \infty$, the space \mathcal{D}_c can be chosen such that $\mathcal{U} = \{[A] \in \mathcal{M}_{X,k}^* : D_q([A_{X_I'}], [A_I]) < \nu\}$, for all I .^{†24}*

²²† The proof of that lemma is an application of the contraction mapping principle.

²³† See §7.2.2 of [8].

²⁴† The distance function D_q is defined by $D_q([A], [B]) = \inf_{h \in \mathcal{G}} \|A - h \cdot B\|_{L^q(X,g)}$.

Proof: We refer to §7.2.2 of [8] generalized to the case of multiple connected sums.

Corollary 1.6.12 *Under the hypotheses of Theorem 1.6.11, there is a homeomorphism $\mathcal{F} : \mathcal{D}_c \longrightarrow \mathcal{V} \subset \mathcal{M}_{X_0,k}^*$ onto an open subset defined by $\mathcal{G}(z) = [\check{A}(z)]$, where $\mathcal{V} \subset \mathcal{M}_{X_0,k}^*$ is the pull-back of the open set $\mathcal{U} \subset \mathcal{M}_{X,k}^*$ of Theorem 1.6.11.*

Neighbourhoods of points at infinity: As we saw in Section 1.4, provided that the base points on X_0 are chosen to be distinct, the gluing maps of Section 1.2 describe neighbourhoods of the bubbling ends of the compactified moduli space $\bar{\mathcal{M}}_{k,X_0}$ away from the diagonal. We extend this result by proving that in fact every boundary point in $\bar{\mathcal{M}}_{k,X_0}$ has a neighbourhood constructible by gluing.

Theorem 1.6.13 *Let the assumptions of Theorem 1.6.11 hold. It then holds that*

- (a) *The approximate gluing map $\mathcal{G} : \mathcal{D} \longrightarrow \mathcal{B}_{X,k}^*$ is a C^∞ embedding.*
- (b) *The extended gluing map $\mathcal{F} : \mathcal{D}_c \longrightarrow \mathcal{U} \subset \mathcal{M}_{X,k}^*$ is a diffeomorphism onto an open subset.*

Proof: (a). The proof uses the same argument required for (b), so it is omitted.

(b). According to Theorem 1.6.11, the map \mathcal{F} is a homeomorphism. Since $\dim \mathcal{D} = \dim \mathcal{M}_{X,k}^*$, it suffices to prove that \mathcal{F} is also an immersion. On the level of connections one sees that there is a C^∞ gluing map which is an immersion and then concludes that the induced map on the level of quotients is a diffeomorphism. The various steps of the proof are analysed in §5.2 of [14]. \square

Corollary 1.6.14 *Under the assumptions of Theorem 2.6.11, the following holds.*

- (a) *The approximate gluing map $\mathcal{G} : \mathcal{D}_c \longrightarrow \mathcal{B}_{X_0,k}$ is a C^∞ embedding.*
- (b) *The extended gluing map $\mathcal{F} : \mathcal{D}_c \longrightarrow \mathcal{U} \subset \mathcal{M}_{X_0,k}$ is a diffeomorphism onto an open subset.*

Theorem 5.12 of [14] asserts that the all ends of the Uhlenbeck compactification can be covered by *gluing neighbourhoods*. We summarise it below for convenience.

Theorem 1.6.15 *Consider the assumptions of Theorem 1.6.11. Let $\{A_a\}$ be a sequence of g_a -ASD connections over $X = \#_{I \in \mathcal{I}} X_I$, the metric g_a being determined by sequences of scales $\{\lambda_{Ia}\}$ converging to 0, points $\{x_{Ia}\}$ converging to $\{x_I\}$ and frames in $\text{Fr} X_0|_{x_{Ia}}$ converging to frames in $\text{Fr} X_0|_{x_I}$. We assume that $\{A_a\}$ converges strongly*

to the bubble-tree limit $(A_I)_{I \in \mathcal{I}}$. For a_0 sufficiently large, there exists a gluing neighbourhood \mathcal{U} such that $[A_a] \in \mathcal{U}$, for all $a \geq a_0$.

Proof: We refer to Theorem 5.12 of [14] and §7.3.1 of [8].

Corollary 1.6.16 *Let the assumptions of Theorem 1.6.11 hold. For every boundary point $(A_0, x_1, \dots, x_l) \in \bar{\mathcal{M}}_{X_0, k}$, there exists a neighbourhood \bar{N} and a parameter space \mathcal{D}_c such that if $\mathcal{N} = \bar{N} \cap \mathcal{M}_{X_0, k}$, then the gluing map $\mathcal{F} : \mathcal{D}_c \rightarrow \mathcal{N} \subset \mathcal{M}_{X_0, k}$ is a diffeomorphism.*

1.7 The space \mathcal{JM}_k

The fact that the gluing map \mathcal{F} of Section 1.6 is a diffeomorphism onto its image combined with the study of a natural stratification of the space $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})$, $0 \leq l \leq k$, of gluing data lead to the construction of a ‘well-structured’ space \mathcal{JM}_k . Our construction essentially follows an inductive argument on l . More precisely, in Section §1.5 we defined a gluing map

$$T_1 : \mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1} \rightarrow \mathcal{JM}_{k,1} = \mathcal{M}_k \coprod \mathcal{M}_{k-1} \times X_0 ,$$

by setting $A \equiv T_1(A_0, \varrho, \pi(p), \lambda_p) = A_0 \#_{p, \lambda_p} I, \dagger^{25}$ where I denotes the standard 1-instanton on S^4 . We recall that if E_{k-1}^* is the bundle E_{k-1} with its zero section removed and $p \in E_{k-1}^*$, we can simply apply the gluing process to obtain a class $[A] \in \mathcal{M}_k$. However, in case that p lies on the zero section of E_{k-1} the picture is different: for $p \in E_{k-1}^0 = \{p \in E_{k-1} : |p| = 0\}$, by construction the map T_1 clearly becomes the identity $Id : \mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} X_0 \rightarrow \mathcal{M}_{k-1} \times X_0$ since the chosen neck-width parameter $\lambda_p = |p|$ vanishes, so roughly speaking we glue a bubble at $x = \pi(p)$. We now recall that

(a). T_1 is a diffeomorphism onto its image, so we identify $\mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1}^*$ with its image $T_1(\mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1}^*)$ in \mathcal{M}_k ,

(b). $\mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1}^*$ is naturally embedded in $\mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1}$.

We proceed to connect the spaces \mathcal{M}_k and $\mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1}$ along $\mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1}^*$ in the apparent way and hence define a space

$$\mathcal{JM}_{k,1} = \mathcal{M}_k \coprod_{T_1} \mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1} \equiv \mathcal{M}_k \coprod_{\mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1}^*} \mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1} .$$

²⁵† Recall that $E_{k-1} = P_{k-1} \times_{SU(2)} \mathbf{H}$.

We note that the stratum $\mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1}$ is of the same homotopy type with the stratum $\mathcal{M}_{k-1} \times X$ of \mathcal{IM}_k since $E_{k-1} = P_{k-1} \times_{SU(2)} \mathbf{H}$ is a quaternionic vector bundle over X_0 . We also observe that $\mathcal{JM}_{k,1}$ and $\mathcal{IM}_{k,1}$ coincide as sets since the union $\mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1} \amalg_{T_1} \mathcal{M}_k$ is realized along the complement of the zero section of E_{k-1} . Finally, the discussion of Section 1.6 actually asserts that the space $\mathcal{JM}_{k,1}$ with its natural topology coincides with the ideal moduli space $\mathcal{IM}_{k,1} = \mathcal{M}_k \amalg \mathcal{M}_{k-1} \times X$ given the topology induced by the notion of weak convergence, in the sense that every weakly convergent sequence of \mathcal{M}_k converges into \mathcal{JM}_k with respect to its identification topology.

For $2 \leq l \leq k$, let us now construct the generalized analogue $\mathcal{JM}_{k,l}$ of $\mathcal{JM}_{k,1}$ by using a family of gluing maps

$$\{\mathcal{F}_l\}_l : \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}) \longrightarrow \mathcal{IM}_{k,l} ,$$

the construction of which emerges explicitly from Corollary 1.6.14. More precisely, let $([A_0], [p_1, \dots, p_l]) \in \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})$. As long as the base points $(\pi(p_1), \dots, \pi(p_l))$ lie away from the diagonal of $Sp^l(X_0)$, one uses centres $x_1 = \pi(p_1), \dots, x_l = \pi(p_l)$ and $y = \infty = n \in S^4$ and scales $\lambda_{p_1}, \dots, \lambda_{p_l}$ to graft the standard 1-instantons $I_1 = I_2 = \dots = I_l = I(0, 1)$ onto A_0 .

In case that the points $(p_1, \dots, p_l) \in Sp^l(E_{k-l})$ project on the diagonal of $Sp^l(X)$, one considers choices of bubble-tree limits $(A_I)_{I \in \mathcal{I}}$ as discussed in §1.6.1 – §1.6.2. The key point is that Corollary 1.6.14 ensures that the gluing map defined in this way gives a diffeomorphism of $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})$ onto its image. We observe now that the gluing parameter space $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})$ has a natural filtration

$$\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}^*) \supset \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})_1 \supset \dots \supset \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})_l ,$$

the q -th ‘level’ $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})_q$ of which is defined by

$$\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})_q = \{[A, (p_1, \dots, p_l)] \in \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}) : \text{at most } q-1 \text{ of the points } (p_1, \dots, p_l) \text{ are of zero length}\} .$$

We consider the corresponding stratification of $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})$, namely

$$\begin{aligned} \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}) &= \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}^*) \amalg \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})_1 \amalg \dots \\ &\dots \amalg \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})_{l-1} \amalg \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})_l . \end{aligned}$$

We recall that if $[p_1, \dots, p_l] \in Sp^l(E_{k-l}^*)$,^{†26} we glue in the familiar way to obtain a class $[A] \in \mathcal{M}_k$. On the other hand, let us assume that $[p_1, \dots, p_l] \in Sp^l(E_{k-l})_1$, with $|p_1| = 0$ say. Then, the scale parameter λ_{p_1} vanishes, so the gluing construction yields a pair $([A], x_1)$ of an *ASD* connection of charge $k-1$ and a curvature-bubble at $x_1 = \pi(p_1)$. Apparently, the procedure follows an inductive pattern, namely if $[p_1, \dots, p_l] \in Sp^l(E_{k-l})_q$, $2 \leq q \leq l$, with $|p_1| = |p_2| = \dots = |p_q| = 0$ say, the scale parameters λ_{p_i} , $1 \leq i \leq q$, vanish. The stratum $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})_q$ can then be identified with the space

$$\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} [Sp^{l-q}(E_{k-l}^*) \times Sp^q(X)] .$$

The gluing map employs the base points of $Sp^{l-q}(E_{k-l}^*)$ to produce an *ASD* connection of charge $k-q$, whilst the points $(x_1, x_2, \dots, x_q) \in Sp^q(X)$ are mapped through the identity forming in this way the centres of concentration for the curvature. The above construction together with Corollary 1.6.14 establish Theorem 1.1.1.

We now work on similar grounds with the case $l = 1$. We observe that for any $1 \leq q \leq l$, we have that

- (a). \mathcal{F}_q is a diffeomorphism, so we may identify $\mathcal{A}_{k-q} \times_{\mathcal{G}_{k-q}} Sp^q(E_{k-q})_{q-1}$ with its image $\mathcal{F}_q(\mathcal{A}_{k-q} \times_{\mathcal{G}_{k-q}} Sp^q(E_{k-q})_{q-1}) \subset \mathcal{M}_{k-q+1} \times Sp^{q+1}(X)$ into $\mathcal{IM}_{k,l}$,
- (b). $\mathcal{A}_{k-q} \times_{\mathcal{G}_{k-q}} Sp^l(E_{k-q})_{q-1}$ is naturally embedded in $\mathcal{A}_{k-q} \times_{\mathcal{G}_{k-q}} Sp^q(E_{k-q})$.

Hence, for any $1 \leq q \leq l$, we proceed to connect the spaces $\mathcal{A}_{k-q} \times_{\mathcal{G}_{k-q}} Sp^q(E_{k-q})$ and $\mathcal{IM}_{k,q-1}$ along the stratum $\mathcal{A}_{k-q} \times_{\mathcal{G}_{k-q}} Sp^q(E_{k-q})_{q-1}$, in order to define a space

$$\mathcal{JM}_{k,l} = \mathcal{M}_k \amalg_{\mathcal{F}_1^*} \mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1} \amalg_{\mathcal{F}_2^1} \mathcal{A}_{k-2} \times_{\mathcal{G}_{k-2}} Sp^2(E_{k-2}) \amalg_{\mathcal{F}_3^2} \dots$$

$$\dots \amalg_{\mathcal{F}_l^q} \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}).$$

where

$$\mathcal{M}_k \amalg_{\mathcal{F}_1^*} \mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1} \equiv \mathcal{M}_k \amalg_{\mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1}^*} \mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1} ,$$

whilst we inductively define

$$\mathcal{A}_{k-q} \times_{\mathcal{G}_{k-q}} Sp^{k-q}(E_{k-q}) \amalg_{\mathcal{F}_{q+1}^q} \mathcal{A}_{k-(q+1)} \times_{\mathcal{G}_{k-(q+1)}} Sp^{q+1}(E_{k-(q+1)}) \equiv$$

$$\mathcal{A}_{k-q} \times_{\mathcal{G}_{k-q}} Sp^{k-q}(E_{k-q}) \amalg_{\mathcal{A}_{k-(q+1)} \times_{\mathcal{G}_{k-(q+1)}} Sp^{q+1}(E_{k-(q+1)})_q} \mathcal{A}_{k-(q+1)} \times_{\mathcal{G}_{k-(q+1)}} Sp^{q+1}(E_{k-(q+1)}).$$

²⁶† Recall that E_{k-l}^* is the bundle $P_{k-l} \times_{SU(2)} \mathbf{H} \xrightarrow{\pi} X_0$ with its zero section removed.

We note that the stratum $\mathcal{A}_{k-q} \times_{\mathcal{G}_{k-q}} Sp^q(E_{k-q})$ is homotopically equivalent to the stratum $\mathcal{M}_{k-q} \times Sp^q(X_0)$ of \mathcal{IM}_k since $E_{k-q} = P_{k-q} \times_{SU(2)} \mathbf{H}$ is a quaternionic vector bundle over X_0 . We also observe that, by construction, $\mathcal{JM}_k \equiv \mathcal{JM}_{k,k}$ and \mathcal{IM}_k coincide as sets. Finally, one checks that by construction the space \mathcal{JM}_k with its natural identification topology coincides with the ideal moduli space \mathcal{IM}_k given the topology induced by the notion of weak convergence.

Remark 1.7.1 To be rigorous as far as the inductive construction of the space \mathcal{JM}_k is concerned, one must show that the transition of the gluing map \mathcal{F}_I from the set $\mathcal{D} = Gl_{x_I} \times B_{x_I}(r_0) \times (0, \lambda_0)$ of gluing data described above to the set

$$\mathcal{D}_c^0 = \{(\varrho_I, y_I, \lambda_I) \in \prod_{I \in \mathcal{I}} Gl_{x_I} \times B_{x_I}(r_0) \times [0, \lambda_0) : \lambda_I = 0 \text{ for some } I\} .$$

is continuous. For the proof of this fact, we refer to Proposition 5.14 of [14] or Proposition 7.2.64 of [8].

Chapter 2

The Palais-Smale structure of \mathcal{B}_k

2.1 Main Results

Let P_k be a principal $SU(2)$ -bundle of 2nd Chern class $c_2(P_k) = k$ over a simply-connected, closed, oriented, 4-dimensional Riemannian manifold X . Let \mathcal{B}_k be the moduli space of gauge equivalence classes of *all* connections on P_k . In this chapter, we study the limiting behaviour of *C-sequences* of \mathcal{B}_k , namely sequences of \mathcal{B}_k with uniformly bounded Yang-Mills functional and functional gradient tending to zero. More precisely, in [41], C. Taubes exposes certain aspects of the behaviour of *C-sequences*, namely sets of *C-sequences* of \mathcal{B}_k are L^2_1 -Sobolev tubular neighbourhoods of finite dimensional sets whose image in \mathcal{B}_k is parametrized via an iterated gluing operation by a finite dimensional space of data which are exhibited in Section 2.3.

Motivated by this result, we employ the approximate gluing maps of Chapter 1 and use a natural stratification of the associated parameter space of gluing data in order to construct a space \mathcal{JB}_k which by definition carries a natural identification topology.

We then show that \mathcal{JB}_k forms a completion for the set of *C-sequences* of \mathcal{B}_k , namely with respect to the identification topology, every *C-sequence* in \mathcal{B}_k have a convergent subsequence with limit point in \mathcal{JB}_k . The reason that makes the study of *C-sequences* particularly interesting originates in Morse theory and is discussed briefly in Section 2.2 - see also Section 0.2.

2.2 The relevant ends of \mathcal{B}_k

Let $\mathcal{YM} : \mathcal{B}_k \longrightarrow \mathbf{R}$ be the *Yang-Mills functional* on \mathcal{B}_k defined by the square of the L^2 -Sobolev norm of the curvature [8], [24]. It is known that elements of the moduli space \mathcal{M}_k of anti-self-dual connections of charge k are absolute minima of the functional \mathcal{YM} [8], [25]. The fact that Donaldson theory [8], [9], [10] related the minimal manifold \mathcal{M}_k to the differential-topology of the underlying 4-manifold X in an intimate way suggests that non-minimal critical points of \mathcal{YM} on \mathcal{B}_k [†] could reveal non-trivial information about the topology of X .

In spite of the fact that Morse theory [31] can provide the framework for examining such a relationship, an obstacle appears because of that \mathcal{B}_k is an infinite-dimensional space, therefore non-compact. It is long known that any ‘naive’[‡] Morse theory will fail to work for \mathcal{YM} .

To ‘recover’ the Morse theory for \mathcal{YM} (in the sense of Theorem 1.1 of [41]), C. Taubes analyzes the restriction of \mathcal{YM} to a countable set of finite-dimensional, non-compact varieties, the so-called *relevant ends* of \mathcal{B}_k with respect to the Yang-Mills functional. Since the relevant ends are ‘narrow’ tubular neighbourhoods of sets of C -sequences in \mathcal{B}_k [41], we convert our initial question to studying the limiting behaviour of those sequences up to convergence of subsequences.

2.2.1 The Palais-Smale condition

Let \mathcal{Z} be a non-compact, Banach manifold and f be a smooth functional on \mathcal{Z} . Given an infinite sequence $\{z_a\} \in \mathcal{Z}$, there are conditions imposed on the real sequence $\{f(z_a)\}$ which when satisfied imply that the hessian data of the set of critical points of f can be used to compute certain topological properties of \mathcal{Z} . The most known such condition is the *Palais-Smale condition* which holds for a pair (\mathcal{Z}, f) if the following is true: each sequence $\{z_a\} \in \mathcal{Z}$ such that (a) $\{f(z_a)\}$ is bounded and (b) the sequence $\{\|\nabla f(z_a)\|\}$ of *gradients* of $\{f(z_a)\}$ tends to zero, has a convergent subsequence in \mathcal{Z} .

Sequences of connections that satisfy Conditions (a) and (b) will henceforth be

[†] The existence of non-minimal critical points of \mathcal{YM} over the 4-sphere is proved in [40] for $k = 1$, whilst for any value of k is proved by Sadun and Segert.

[‡] To use C. Taubes’ expression for $SU(2)$ -equivariant Morse theory.

referred to as *C-sequences*. We note that the Palais-Smale condition is a compactification condition and is satisfied automatically whenever \mathcal{Z} is compact. Moreover, C-sequences in a compact \mathcal{Z} converge to critical points of f . The condition can fail for an infinite-dimensional manifold \mathcal{Z} because of the non-compactness.

Therefore, when dealing with a non-compact variational problem we naturally focus on C-sequences and investigate their limiting behaviour according to the following scheme.

- (a) Determine the *non-compact ends* of \mathcal{Z} which are ‘relevant’ to the functional f , i.e. the sets of points with bounded *energy* and functional gradient tending to zero,
- (b) *Analyze* the behaviour of f on the relevant ends.

Definition 2.2.1 Let E be a fixed energy for f and δ be a small positive number. The (E, δ) -end of \mathcal{Z} relevant to f is defined by

$$\text{Crit}(E, \delta) = \{z \in \mathcal{Z} : |f(z) - E| < \delta \text{ and } \|\nabla f_z\| < \delta\},$$

where $| \cdot |$ is the standard norm in \mathbf{R} and $\| \cdot \|$ is a norm on the tangent space $T\mathcal{Z}$ which can be weaker than the canonical norm.

The key point is that in non-compact variational problems, a deep understanding of the topological properties of the relevant ends can contribute to answer what topological features of the manifold \mathcal{Z} are represented by critical points of f . The algebraic-topological framework required to analyze f in the non-compact ends introduces a set $\text{Crit}(E, \delta)^- = \{z \in \text{Crit}(E, \delta) : f(z) < E\}$ and for given $\delta > \delta' > 0$ considers the natural inclusion of pairs $i : (\text{Crit}(E, \delta'), \text{Crit}(E, \delta')^-) \longrightarrow (\text{Crit}(E, \delta), \text{Crit}(E, \delta)^-)$.

As it is shown in [41], it is the behaviour of the map i on the relative homotopy groups of the above pairs which determine whether the ends $\text{Crit}(E, \delta)$ contribute to the Morse theory of f on \mathcal{Z} . As this construction is beyond the scope of this text, we refer to Section 2 of [41] for a non-technical description.

The case $(\mathcal{Z}, f) = (\mathcal{B}_k, \mathcal{YM})$: Let us consider the above situation for the Yang-Mills functional \mathcal{YM} on the moduli space \mathcal{B}_k of all connections of charge k . As already mentioned, the Palais-Smale condition fails for the Yang-Mills functional since \mathcal{B}_k is infinite-dimensional. We aim to construct a space \mathcal{JB}_k in which every C-sequence in \mathcal{B}_k has a convergent subsequence. The scheme proposed above, leads us to define

the relevant ends of \mathcal{B}_k determined by the functional \mathcal{YM} . Therefore, we fix positive Yang-Mills energy E and consider a small positive number δ .

The (E, δ) -end of \mathcal{B}_k relevant to \mathcal{YM} is then defined by

$$\text{Crit}(E, \delta) = \{[A] \in \mathcal{B}_k : |\mathcal{YM}(A) - E| < \delta \text{ and } \|\nabla \mathcal{YM}_A\| < \delta\},$$

where the norm $\|\cdot\|$ is defined in §2 of Section 3 of [41]. These are the ‘critical ends’ of \mathcal{B}_k in the sense that sequences in $\text{Crit}(E, \delta)$ may tend to infinity without converging to critical points. Let $[A_a]$ be a sequence in $\text{Crit}(E, \delta)$, i.e. $|\mathcal{YM}(A_a) - E| < \delta$ and $\|\nabla \mathcal{YM}_{A_a}\| < \delta$. If we let $\delta \rightarrow 0$, then $[A_a]$ actually becomes a C -sequence. Hence, the set $\text{Crit}(E, \delta)$ forms a δ -tubular neighbourhood of C -sequences \mathcal{B}_k of energy E . This is precisely the reason that we study these sequences: to analyze the structure of the non-compact ends $\text{Crit}(E, \delta)$ of \mathcal{B}_k .

The main step in the analysis is the study of the limiting behaviour of sequences $\{A_a \in \text{Crit}(E, 2^{-a}\delta)\}_a$. K. Uhlenbeck’s *weak compactness theorem* [15] is the main tool in this analysis, the typical pathology being the convergence to a critical point of the \mathcal{YM} functional off a finite set of points of the underlying 4-manifold X . However, the charge of the ‘limit orbit’ may not be equal to the charge of the reference sequence. Thus, to a first approximation, the relevant ends of \mathcal{B}_k form a disjoint union of sets that are parametrized by finite sets of points in X and by critical points of the \mathcal{YM} functional that are orbits of connections on some *allowed* set of principal $SU(2)$ -bundles over X .

As it will soon become evident, one certainly needs additional parameters to describe the structure of the relevant ends. More precisely, as we shall see in Section 2.4, as the index of $\{A_a \in \text{Crit}(E, 2^{-a}\delta)\}_a$ increases the term A_a gets closer to a ‘point’ which lies in the image of an iterated gluing map described in Section 2.3. This parametrization uses topological data of the 4-manifold X , namely configurations of points on fiber bundles over X and critical points of the Yang-Mills functional on the 4-sphere. Moreover, the total number of data involved in the construction is bounded by a linear function of the energy E .

Non-compact variational problems: The Yang-Mills functional \mathcal{YM} exhibits phenomena that are also found in other (elliptic) variational problems, where the functional f fails to satisfy the Palais-Smale condition on a Banach manifold \mathcal{Z} . An example is

provided for instance by the *harmonic map problem*, where one studies the *energy functional* $E(\phi) = \int_X |\nabla^2 \phi| \text{vol} X$ on the Banach manifold $\mathcal{Z} = \{\phi : \Sigma \rightarrow (X, \omega)\}$ of harmonic maps from a Riemann surface Σ to a Kahler manifold (X, ω) - see [37], [38].

In all these problems, the behaviour of a sequence $\{A_a \in \text{Crit}(E, 2^{-a}\delta)\}_a \subset \mathcal{Z}$ is parametrized by data that are composed of (a) a solution to the Euler-Lagrange equation of the particular variational problem on the reference Riemannian manifold X and a set of a finite number of 4-tuples composed of (b) a properly *centred* solution to an appropriate conformally invariant set of equations on the standard sphere of the same dimension with X , (c) a conformal dilation of the standard sphere, (d) a *gluing parameter*³ and (e) a base point on X .

Let us note that the data (a) – (e) are usually required to obey certain energy constraints, namely the value of the given functional, evaluated on a solution in (a) plus the sum of the values of an appropriate conformally invariant functional, evaluated on the solutions in (b) is required to equal a fixed energy E .

The procedure one follows then is schematically described below: we take the solution of (a) on X and by using the gluing parameters in (d) we glue up the sphere solutions in (b) by identifying a neighbourhood of the point in (e) with a neighbourhood of the north pole of S^4 , after conformally dilating them using elements in (c). The data (b) – (e) are usually called *gluing data*.

2.3 An iterated gluing scheme

An Overview: Let P_k be an $SU(2)$ bundle over a simply-connected 4-manifold X and \mathcal{B}_k be the moduli space of all connections on P_k . For fixed Yang-Mills energy $E > 0$, it is shown in [41] that the relevant end $\text{Crit}(E, \delta)$ is an $SU(2)$ -invariant, tubular neighbourhood of a set $\mathcal{N}(E) \subset \mathcal{B}_k$ which can be parametrized by data defined *inductively* as follows:

$$e = [A_0, [q_a, e_a]] , \quad 1 \leq a < \infty . \quad (2.1)$$

³† To be able to ‘add’ elements of a non-linear space of sections of an appropriate connected sum fiber bundle, we need to choose a local identification of fibers over the individual summands. For each point in (e), a gluing parameter provides such an identification map.

In Equation (2.1), A_0 is a smooth connection on an $SU(2)$ bundle $P_0 \rightarrow X$ of 2nd Chern class $k_0 \leq k$. Moreover, A_0 is a solution to the Yang-Mills equations over X . Henceforth, A_0 will be referred to as the *background* connection. The set $\{q_a\}$ consists of points in $P_0 \otimes FrX$, where $\pi : FrX \rightarrow X$ is the bundle of orthonormal frames over X . We recall that if $(p, v) \in P_0 \otimes FrX$, then the frame v defines a normal geodesic system for the base point $x = \pi(p)$. Let us stress here that the points $\{q_a\} \in X$ are required to be distinct.

In Equation (2.1), e_a is a member of a set that is parametrized as follows:

$$e_a = (\lambda_a, [p_a, A_a, [q_{a,b}, e_{a,b}]]) , \quad 1 \leq b < \infty . \quad (2.2)$$

We recall that $\lambda_a \in (0, 1]$ is a scale parameter which measures the width of the neck of the connected sum of X with the 4-sphere S^4 . In Equation (2.2), p_a is a point on the fiber over the north pole in S^4 of an $SU(2)$ -bundle P_a of 2nd Chern class $k_a \leq k - k_0$. We should mention that A_a is a suitably *centred* solution of the Yang-Mills equation on S^4 in the sense of §1.3.3.

As it is already known from Chapter 1, the idea is to apply a cut and paste operation in order to graft the instantons A_a onto the background connection A_0 , using the data $\{q_a, \lambda_a, p_a\}$. To define each point $q_{a,b}$, we fix a conformal diffeomorphism Φ , namely the inverse of the stereographic projection. Then, $\{q_{a,b}\}$ is a set of distinct points of Φ^*P_a and $e_{a,b}$ is an element that is defined inductively using Equation (2.2).

The process continues, to produce a finite set $\{e_a, e_{a,b}, \dots, e_{a,b,\dots,z}\}$ and for the point $e_{a,b,\dots,z}$ it holds that $e_{a,b,\dots,z} = (\lambda_{a,b,\dots,z}, [p_{a,b,\dots,z}, A_{a,b,\dots,z}])$, i.e. $e_{a,b,\dots,z}$ forms the last step in the gluing construction and the connection $A_{a,b,\dots,z}$ cannot be of zero curvature. The gluing data chosen above are bound to satisfy the following sum-rules:

$$k_0 + \sum_a (k_a + \sum_b (k_{a,b} + \dots + \sum_z k_{a,b,\dots,z}) \dots) = k \quad (2.3)$$

$$\mathcal{YM}(A_0) + \sum_a (\mathcal{YM}(A_a) + \sum_b (\mathcal{YM}(A_{a,b}) + \dots + \sum_z \mathcal{YM}(A_{a,b,\dots,z}) \dots) = E \quad (2.4)$$

These constraints plus the gap theorems in [43] imply that the total number of terms appearing in the construction is bounded by a linear function of E .^{†4}

^{†4} Roughly speaking, we keep gluing until we *exhaust* the available energy E .

2.3.1 The gluing method

In [41], [43], C. Taubes employs an iterated gluing construction to define a gluing map

$$T : [\mathcal{A}(P_0) \times_{\mathcal{G}(P_0)} (P_0 \otimes FrX \otimes (0, 1])] \times_{SU(2) \times SO(4)} [\mathcal{A}(P_1) \times_{\mathcal{G}(P_1)} P_1|_n] \longrightarrow \mathcal{B}_k ,$$

where \mathcal{B}_k is the moduli space of connections on an $SU(2)$ -bundle of 2nd Chern class $k = k_0 + k_1$. We survey the construction of the map T , the detailed exposition of which can be found in Section 2 of [41]. For the convenience of the reader we have followed closely the notation of [41].

More precisely, the group $SO(4)$ acts on FrX and $\mathcal{A}(P_1) \times_{\mathcal{G}(P_1)} P_1|_n$ by $(u, f) \rightarrow f \cdot u^{-1}$ and $(u, [A, p]) \rightarrow [u^{-1}A, p]$, respectively. We note that the inverse $\Phi : \mathbf{R}^4 \rightarrow S^4 \setminus \{n\}$ of the stereographic projection from the north pole n , provides a homomorphism of $SO(4)$ into the group of conformal diffeomorphisms of S^4 which fix n . A choice $v \in FrX$ defines a unique Gaussian coordinate system Φ_x^{-1} for the base point $\pi(v) = x \in X$. We note that Φ_x^{-1} identifies $B_x(1) \subset X$ with $B_0(1) \subset \mathbf{R}^4$ and $\Phi \circ \Phi_x^{-1}$ identifies $B_x(1)$ with the ball of radius π in S^4 and also x with the south pole s .

We choose a gluing parameter $\varrho : P_0|_x \rightarrow P_1|_n$ which we extend over $\mathcal{U} = B_x(1) \setminus x$ as follows. A point $(A_0, p_0) \in \mathcal{A}(P_0) \times_{\mathcal{G}(P_0)} P_0$ determines a section $s(A_0, p_0)$ of P_0 over $B_x(1)$ through the A_0 -parallel transport of p_0 along the short radial geodesics through x . This section clearly defines a trivialization of P_0 over $B_x(1)$. Similarly, a point $(A_1, p_1) \in \mathcal{A}(P_1) \times_{\mathcal{G}(P_1)} P_1$, defines a section $s(A_1, p_1)$ of P_1 , thus a trivialization of P_1 over $S^4 \setminus \{n\}$. We identify these trivializations in order to obtain a principal $SU(2)$ bundle $P = P_0 \# P_1$ of charge $k_0 + k_1$ over the connected sum $Y = X \# S^4$. Let us denote P by $P(\omega)$ to stress its dependence upon the gluing data

$$\omega = [([A_0, p_0], f, \lambda), [A_1, p_1]] \in \mathcal{A}(P_0) \times_{\mathcal{G}(P_0)} (P_0 \otimes FrX \times (0, 1]) \times_{SU(2) \times SO(4)} [\mathcal{A}(P_1) \times_{\mathcal{G}(P_1)} P_1|_n] .$$

Next, we define a family of connections $\{A(\omega, t) \in B(P(\omega)) : t \in (0, 1/8]\}$ for a point ω as above. More precisely, let $n_\lambda(y) = \{n(|y|/\lambda) : y \in \mathbf{R}^4, \lambda \in (0, 1]\}$ be a family of smooth cut-off functions with values in $[0, 1]$. We require that $n(z) = 0$ if $z < 1/2$ and $n(z) = 1$ if $z > 1$. Given $t \in (0, 1/8]$, we define a family of connections $\{A(\omega, t) : t \in (0, 1/8]\}$ on $P(\omega)_{B_1(x) \setminus x}$, by the formula

$$A(\omega, t) = C + n_{8\lambda} \phi(A_0)(h_0)^* A_0 + (1 - n_{\sqrt{\lambda}}) \lambda^* \phi(A_1)(h_1)^* A_1 ,$$

where C is the trivial product connection on $P(\omega)$.^{†5} Let us note here that $P(\omega)$ was constructed in such a way that it has a canonical product structure over \mathcal{U} .

On the set $Z_1 = \{p \in S^4 : \text{dist}(p, x) > \sqrt{\lambda}/t\}$, $P(\omega)$ is canonically identified with P_0 and a family of connections $\{A(\omega, t) : t \in (0, 1/8]\}$ on $P(\omega)|_{Z_1}$ is given by setting $A(\omega, t) = A_0$. Similarly, on $Z_2 = \{p \in B_1(x) : \text{dist}(p, x) < \lambda/2\}$, $P(\omega)$ is identified with λ^*P_1 and a family of connections $\{A(\omega, t) : t \in (0, 1/8]\}$ on $P(\omega)|_{Z_2}$ is given by setting $A(\omega, t) = \lambda^*A_1$. These three families agree where the domains of the definition overlap, therefore they define a smooth family of glued connections on $P(\omega)$.

Let us now consider l principal $SU(2)$ bundles $P_a \rightarrow S^4$ of 2nd Chern class k_a , $a = 1, \dots, l$. The map T can be generalized in a straightforward way to give a map

$$\Theta_l(t) : \mathcal{A}(P_0) \times_{\mathcal{G}(P_0)} Sp^l(P_0 \otimes FrX \times (0, 1]) \times_{SU(2) \times SO(4)} \prod (\mathcal{A}(P_a) \times_{\mathcal{G}(P_a)} (P_a|_n))^* \rightarrow \mathcal{B}_k,$$

where $k = \sum_{a=0}^l k_a$. The ‘ $*$ ’ on the data aims to signify that the base points $\{x_a\}$ in the product $Sp^l(P_0 \otimes FrX \times (0, 1])$ must be distinct and that the set of scales $\{\lambda\}$ must obey the constraint

$$\frac{\sqrt{\lambda_a}}{t} + \frac{\sqrt{\lambda_b}}{t} \leq \frac{1}{4} \text{dist}(x_a, x_b).$$

When $X = \mathbf{R}^4$, one can use a fixed global Euclidean coordinate system and use the gluing operation to construct the equivalent of the map $\Theta(t)_l$. More precisely, assuming the same restrictions as in $\Theta_l(t)$ one defines a gluing map

$$\Theta_l(0, t) : \mathcal{A}(P_0) \times_{\mathcal{G}(P_0)} Sp^l((P_0 \times (0, 1]) \times_{SU(2)} \prod (\mathcal{A}(P_a) \times_{\mathcal{G}(P_a)} (P_a|_n))^*) \rightarrow \mathcal{B}_k.$$

We also observe that if $X = S^4$ and consider $P_0 \rightarrow S^4$ to be a principal $SU(2)$ -bundle, then $\Theta_l(0, t)$ induces a gluing map

$$\Theta_l(1, t) : \mathcal{A}(P_0) \times_{\mathcal{G}(P_0)} Sp^l(P_0|_{S^4 \setminus \{n\}} \times (0, 1]) \times_{SU(2)} \prod (\mathcal{A}(P_a) \times_{\mathcal{G}(P_a)} (P_a|_n))^* \rightarrow \mathcal{B}_k.$$

The map $\Theta_l(1, t)$ is obtained from $\Theta_l(0, t)$ via the inverse Φ of the stereographic projection. The introduction of $\Theta_l(1, t)$ facilitates certain apriori estimates which are

^{5†} Let us note that a conformal dilation of S^4 with source the south pole and sink the north pole is equivalent to a rescaling of \mathbf{R}^4 via the inverse Φ of the stereographic projection. Under this identification, λ denotes both the number in $(0, \infty)$ and the conformal dilation of S^4 .

required when the gluing map is iterated as in 2.3.2. If the base point on S^4 is the north pole n , then $\Theta(1, t)$ lifts to a map

$$\Theta_l(1, t)' : [P_{0|n} \times \mathcal{A}(P_0) \times_{\mathcal{G}(P_0)} Sp^l(P_{0|S^4 \setminus n} \times (0, 1]) \times_{SU(2)} \prod (\mathcal{A}(P_a) \times_{\mathcal{G}(P_a)} (P_{a|n}))^* \longrightarrow \mathcal{B}_k.$$

2.3.2 An iterated gluing map

To analyze the limiting behaviour of C -sequences in \mathcal{B}_k we need to define a gluing map \mathcal{R}_l whose domain $\{e_a, e_{a,b}, \dots, e_{a,b,\dots,z}\}$, defined inductively by Equation (2.2), is required to satisfy various constraints. Namely, let P_k be an $SU(2)$ bundle over the 4-manifold X and $E > 0$ be a fixed Yang-Mills energy.

Constraint 1: We define $\mathcal{E}(P, E)$ to be the set $\mathcal{E} = \{e \equiv [A_0, [q_a, e_a]]\}$, where the data $\{e\}$ is defined inductively by Equations (2.2)-(2.4) but with the following restriction. Let us denote the data $\{e_a, e_{a,b}, \dots, e_{a,b,\dots,z}\}$ by $e \dots \equiv (\lambda \dots, [p \dots, A \dots, [q \dots z, e \dots z]])$. If A is a smooth instanton on S^4 , we define the *centre* $y[A]$ and *scale* $\lambda[A]$ of A by⁶

$$y[A] \equiv \mathcal{YM}(A)^{-1} \int y |\Phi^* F_A| d^4 y \in \mathbf{R}^4, \quad \lambda[A]^2 \equiv \frac{16}{\kappa_0^2} \int |y - y[A]|^2 |\Phi^* F_A|^2 d^4 y,$$

and require that the instantons of \mathcal{E} satisfy $y[A] = 0$ and $\lambda[A] = 1$. We also require that

$$y[A]^2 \cdot \mathcal{YM}(A \dots) + \kappa_0^2 \frac{\lambda[A]^2}{16} < \frac{\kappa_0^2}{16}, \quad (2.5)$$

where the constant $\kappa_0 \in (0, 1/4)$ is defined in Lemma 9.1 of [41]. From the data $e \in \mathcal{E}$, let us denote by \underline{e} the set $\underline{e} = \{[A_0], [A_a], \dots, [A_{a,b,\dots,z}]\}$.

Constraint 2: Given $t \in (0, 1/8]$, we define a domain $\mathcal{E}(P, E, t) \subset \mathcal{E}(P, E)$ as follows. We require of $e \in \mathcal{E}(P, E, t)$ that the data $\{\lambda_{a,\dots,z}, \pi(q_{a,\dots,z})\}$ obey

$$\lambda_{\dots,z} < t \quad \text{and} \quad \frac{\sqrt{\lambda_{\dots,z}}}{t} + \frac{\sqrt{\lambda_{\dots,w}}}{t} \leq \frac{1}{4} \text{dist}(\pi_{q_{\dots,z}}, \pi_{q_{\dots,w}}) \quad (2.6)$$

Here, $\text{dist}(\cdot, \cdot)$ is the geodesic distance on X for $\{\lambda_{\dots,z}, \pi(q_{\dots,z})\} \equiv \{\lambda_z, \pi(q_z)\}$, otherwise it is the Euclidean distance on \mathbf{R}^4 .

⁶† Compare with the definition of §1.3.3.

Constraint 3: We require of $e \in \mathcal{E}(P, E, t)$ that $\{([A_a], \lambda_a, \dots, z, \pi(q_a, \dots, z))\}$ obey the inequality

$$\int_{\text{dist}(\cdot, x) < \frac{\sqrt{\lambda}}{t}} \text{dist}(\cdot, x)^{-2} |F_A|^2 < \left(\kappa_0 \frac{t^2}{4} \sqrt{\lambda}\right)^2. \quad (2.7)$$

The gluing map: Given $t \in (0, 1/8]$, we define a map $\mathcal{R}_t : \mathcal{E}(P, E, t) \longrightarrow \mathcal{B}(P)$ by

$$\mathcal{R}_t(e) = \Theta_t(t)([A_0, [q_a, \lambda_a, [\Theta_t(1, t)'] (p_a, A_a, [q_{a,b}, \lambda_{a,b}, [\Theta(1, t)'] (\dots [p_{a,b,\dots,z}, A_{a,b,\dots,z}] \dots) \dots$$

where $\Theta_t(1, t)'(\cdot)$ is any connection with gauge orbit $\Theta_t(1, t)(\cdot)$, as in 2.3.1.

The gluing map \mathcal{R}_t is iterated in the sense that the connections on S^4 that we attach to the base manifold X are themselves formed from a gluing construction on S^4 - see construction of map $\Theta_t(1, t)$. We recall that a similar iterative scheme was used in Chapter 1 to prove the existence of a universal gluing map.

2.4 The limiting behaviour of C -sequences in \mathcal{B}_k

In this section, we expose the limiting behaviour of C -sequences of \mathcal{B}_k following the results of [41], [43]. As we have already explained in Section 2.2, given a Yang-Mills energy E , one first determines the relevant ends $\text{Crit}(E, \delta)$ in \mathcal{B}_k and uses Uhlenbeck's weak compactness theorem [45] and removability of singularities theorem [44] to analyze the behaviour of sequences in $\text{Crit}(E, \delta)$. The key result is proved in Section 4 of [43] and is summarized for convenience below.

Proposition 2.4.1 *Let P_k be an $SU(2)$ -bundle over X . Let $\{[A_i]\}$ be a C -sequence of orbits in \mathcal{B}_k . Then, there exists (a) a set of points $\Omega = \{x_m\} \subset X$, (b) a principal $SU(2)$ -bundle P_0 over X of 2nd Chern class $k_0 \leq k$, (c) a smooth connection A_0 on P_0 such that $\mathcal{YM}(A_0) \leq E$ and which solves the Yang-Mills equations on X , (d) a sequence of bundle isomorphisms $\{g_i\} \in L_3^2(\text{Iso}(P|_Y, P_0|_Y))$, where $Y = X \setminus \Omega$, (e) a set of l pairs $\{(P_a, A_a)\}_a$, where each P_a is a principal $SU(2)$ -bundle of class k_a and A_a is a smooth connection on P_a which solves the Yang-Mills equations on S^4 . Then, there exists a subsequence of $\{g_i A_i\}$ which converges to A_0 in the L_1^2 -Sobolev topology on Y . Furthermore, the following sum-rules are satisfied:*

$$k_0 + \sum_a k_a = k \quad \text{and} \quad \mathcal{YM}(A_0) + \sum_a \mathcal{YM}(A_a) = E.$$

The set Ω is characterised in Proposition 4.5 of [43], namely there exists a constant $\kappa > 0$ such that if for an open set $U \subset X$ it holds that $\liminf |F_{A_i}|_{L^2_U} < \kappa$, then $U \cap \Omega = \emptyset$. In Proposition 2.4.1, the failure of convergence of the sequence $\{g_j A_j\}$ at the points of Ω is due to the ‘bubbling’ of the curvature. The data $\{(P_a, A_a)\}$ is obtained by studying precisely this behaviour.

The forthcoming arguments concerning the asymptotical behaviour of C -sequences require the introduction of a particular class of sequences of gluing data in $\mathcal{E}(P, E)$.

Definition 2.4.2 A *limit sequence* of gluing data in \mathcal{E} is a sequence $\{e[i]\}$ whose dependence on the index i has the following properties.

- (a). The induced sequence of 2nd Chern classes is such that $\{k_0[i], k \dots [i]\} \equiv \{k_0, k\}$, i.e. it is independent of i .
- (b). The induced sequence of orbits is such that $\{[A_0[i]] \in \mathcal{B}_{k_0}, [A \dots [i]] \in \mathcal{B}(k \dots)\} \equiv \{[A_0], [A \dots [i]]\}$, i.e. it is independent of i .
- (c). Each sequence $\{\lambda \dots [i]\}$ of scales is monotonically decreasing to zero.
- (d). Each sequence $\{\pi(q \dots [i])\}$ is convergent.

Condition (a) asserts that the terms of the induced sequence $\{P_0[i]\}$ of $SU(2)$ -bundles over X lie in the same isomorphic class. Similarly, the terms of the induced sequence $\{P \dots [i]\}$ of $SU(2)$ -bundles over S^4 are isomorphic. Condition (b) asserts that the terms of the induced sequence $\{A_0[i]\}$ of background connections over X lie in the same gauge equivalence class. Similarly, the induced sequence $\{A \dots [i]\}$ of connections over S^4 are gauge equivalent.

The following proposition is a refinement of Proposition 2.4.1 and describes what C -sequences of \mathcal{B}_k look like asymptotically.

Proposition 2.4.3 *Let us consider the assumptions of Proposition 2.4.1. Given $t \in (0, 1/8]$, there exists (a) a subsequence of $\{[A_j]\}$, henceforth relabelled as $\{[A_j]\}$ and (b) a limit sequence $\{e[i]\} \in \mathcal{E}(P, E, t)$, with the following significance. For given $\epsilon > 0$ there exists $j(\epsilon)$ such that for all $j > j(\epsilon)$, there exists a connection A on P with gauge orbit $[A] = \mathcal{R}_l(e[j])$ and with $\|[A_j] - \mathcal{R}_l(e[j])\|_{L^2_1} < \epsilon$.*

Proof: We refer to Proposition 5.3 of [41].

Let us note that Proposition 2.4.3 is stated in [41] with respect to the norm $\|\cdot\|_A$ introduced in §2 of the same paper. However, the equivalence of $\|\cdot\|_A$ to the L^2_1 -Sobolev

norm is proved in §2 of [43]. Morally speaking, Proposition 2.4.3 asserts that given a sequence $\{[A_j]\}$ in \mathcal{B}_k which approaches the relevant ends, there exists a limit sequence $(e[i])$ of gluing data and an index l such that the sequence $\{[G_j]\} = \mathcal{R}_l(e[i])$ of glued connections is within zero L_1^2 -Sobolev distance from $\{[A_j]\}$. Consequently, the two sequences have the same limit, therefore to understand the structure of $\text{Crit}(E, \delta)$ it suffices to study the limiting behaviour of $\{[G_j]\}$.

For convenience, we state two results of [41] concerning the structure of the relevant ends $\text{Crit}(E, \delta)$.

Proposition 2.4.4 *Let $P \rightarrow X$, $E > 0$, $\epsilon > 0$ and $t \in (0, 1/8]$ be as above. Then, there exists $\delta > 0$ such that, given $E' \leq E$ and $c \in \text{Crit}(E', \delta)$ there exists E'' , a point $e \in \mathcal{E}(P, E', t)$ and connections A and A' such that $[A] = c$ and $[A'] = \mathcal{R}_l(e)$ and such that $\|[A'] - A\| < \epsilon$. Conversely, for $e \in \mathcal{E}(P, E, t)$, we have that $\mathcal{R}_l(e) \in \text{Crit}(E, z_0 \cdot E \sqrt{\underline{\lambda}} \cdot t)$, where $\underline{\lambda} \equiv \sup\{\lambda \dots\}$, the supremum being over all scale parameters.*

Proof: We refer to Proposition 5.4 of [41].

Proposition 2.4.5 *Let $P_k \rightarrow X$, $P_{k_0} \rightarrow X$, $E > 0$, $\epsilon > 0$ and $t \in (0, 1/8]$ be as above. We define the spaces*

$$N(E, \epsilon) \equiv \{\mathcal{R}_l(e) : e \in \mathcal{E}(P, E, t \equiv \epsilon)\} ,$$

$$\bar{N}(E, \epsilon) \equiv \{[A] : \|A' - A\| < \epsilon, \text{ for some } [A'] \in N(E, \epsilon)\} ,$$

$$N([A_0], E, \epsilon) \equiv \{\mathcal{R}_l(e) \in N(E, \epsilon) : e \equiv [A_0, [q_a, e_a]] \in \mathcal{E}(P, E, t \equiv \epsilon)\} ,$$

$$\bar{N}([A_0], E, \epsilon) \equiv \{[A] \in N(E, \epsilon) : \|A' - A\| < \epsilon, \text{ for some } [A'] \in N([A_0], E, \epsilon)\} .$$

Then, there exists $\epsilon_0 > 0$ and given $\epsilon \in (0, \epsilon_0]$ there exists $\epsilon_1 > 0$ such that $\text{Crit}(E, \epsilon_1) \subset N(E, \epsilon)$ and $N(E, \epsilon_1) \subset \text{Crit}(E, \epsilon)$.

Proof: We refer to Proposition 5.6 of [41].

Conclusion: Propositions 2.4.1 and 2.4.3-2.4.5 describe the (stratified) structure of the relevant ends of \mathcal{B}_k , namely the ends form a disjoint union of $SU(2)$ -invariant sets that consist of connections which are close, with respect to the L_1^2 -Sobolev norm, to connections which are gluings of a finite number of centred instantons on S^4 to a background connection of the reference 4-manifold X . In Section 2.5, we exploit this fact to construct a space \mathcal{JB}_k such that every C -sequence of \mathcal{B}_k converges (modulo a blow-up phenomenon) to a limit point in \mathcal{JB}_k .

2.5 The space \mathcal{JB}_k

We use the approximate gluing construction of Chapter 1 in order to define the ‘limit space’ \mathcal{JB}_k mentioned in the Conclusion of Section 2.4. We adapt the scheme of Section 1.7 noticing that the technology we require is less sophisticated than the one we employed there to construct the space \mathcal{JM}_k . This is simply due to the independence of the approximate gluing construction upon numerous energy estimates which for \mathcal{M}_k ensured that the gluing maps of Section 1.6 yield diffeomorphisms onto their image.

Let \mathcal{A}_k be the space of all connections on a principal $SU(2)$ bundle P_k over a simply-connected, Riemannian 4-manifold X . Let also $\pi : E_{k-1} \rightarrow X$ be the Hermitian quaternionic bundle $E_{k-1} = P_{k-1} \times_{SU(2)} \mathbf{H}$. We define a map

$$T_1 : \mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1} \longrightarrow \mathcal{IB}_{k,1} = \mathcal{B}_k \coprod \mathcal{B}_{k-1} \times X_0 ,$$

by $A = T_1(A_0, \varrho, p, \lambda_p) \equiv A_0 \#_{\pi(p), \lambda_p} I$, copying the gluing construction of Section 1.6. We recall that if E_{k-1}^* is the bundle E_{k-1} with its zero section removed and $p \in E_{k-1}^*$, then, we glue a copy of the standard 1-instanton I on S^4 to obtain a class $[A] \in \mathcal{B}_k$. In case that $p \in E_{k-1}^0 = \{p \in E_{k-1} : \lambda_p = 0\}$, by construction the map T_1 becomes the identity $Id : \mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} X_0 \longrightarrow \mathcal{B}_{k-1} \times X_0$ since the chosen scale parameter λ_p vanishes.

We now observe that (a) T_1 is an C^∞ embedding, so we may identify $\mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1}^*$ with its image $T_1(\mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1}^*)$ in \mathcal{B}_k and (b) $\mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1}^*$ is naturally embedded in $\mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1}$. We connect \mathcal{B}_k with $\mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1}$ along $\mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1}^*$ in order to define the space

$$\mathcal{JB}_{k,1} = \mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1} \coprod_{T_1} \mathcal{B}_k \equiv \mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1} \coprod_{\mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1}^*} \mathcal{B}_k .$$

We note that by construction $\mathcal{IB}_{k,1}$ and $\mathcal{JB}_{k,1}$ coincide as sets. We proceed to define the generalized analogue of the space $\mathcal{JB}_{k,1}$. We introduce a family of gluing maps $\mathcal{F}_l : \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}) \longrightarrow \mathcal{JB}_{k,l}$, the construction of which emerges explicitly from the approximate map \mathcal{G} of Section 1.6. Moreover, Corollary 1.6.14 ensures that the gluing map defined in this way is a C^∞ embedding.

Let us consider the parameter space $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})$ and observe that it has a natural filtration

$$\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}^*) \supset \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})_1 \supset \dots \supset \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})_l ,$$

where the q -th level $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})_q$, $0 \leq q \leq l$, is defined by

$\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})_q = \{[A, (p_1, \dots, p_l)] \in \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}) : \text{at most } q-1 \text{ of the points } (p_1, \dots, p_l) \text{ are of zero length}\}.$

We consider the corresponding stratification of $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})$, namely

$$\begin{aligned} \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}) &= \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}^*) \amalg \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})_1 \amalg \dots \\ &\dots \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})_{l-1} \amalg \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})_l . \end{aligned}$$

We recall that if $[p_1, \dots, p_l] \in Sp^l(E_{k-l}^*)$,^{†7} we glue in the familiar way to obtain a class $[A] \in \mathcal{B}_k$. In case that $[p_1, \dots, p_l] \in Sp^l(E_{k-l})_1$, with $|p_1| = 0$ say, the scale parameter λ_{p_1} vanishes, therefore the gluing construction yields a pair $([A], x_1)$ of an *ASD* connection of charge $k-1$ and a curvature-bubble at $x_1 = \pi(p_1)$. Apparently, our construction follows an inductive argument on l : if $[p_1, \dots, p_l] \in Sp^l(E_{k-l})_q$, $2 \leq q \leq l$, with $|p_1| = |p_2| = \dots = |p_q| = 0$ say, the scale parameters λ_{p_i} , $1 \leq i \leq q$, vanish. The stratum $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l})_q$ can be then identified with the space

$$\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} [Sp^{l-q}(E_{k-l}^*) \times Sp^q(X)] .$$

The gluing construction uses the base points in $Sp^{l-q}(E_{k-l}^*)$ to produce an *ASD* connection of charge $k-q$, whilst the points $(x_1, x_2, \dots, x_q) \in Sp^q(X)$ are mapped through the Identity forming the concentration centres for the curvature. The above construction together with Corollary 1.6.14 establish a map

$$\mathcal{F}_l : \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}) \longrightarrow \mathcal{JB}_{k,l} ,$$

with the properties (a) \mathcal{F}_l is defined for all $1 \leq l \leq k$, (b) \mathcal{F}_l is a C^∞ embedding.

We can now copy the argument of the case $l = 1$ and observe that for any $1 \leq q \leq l$, it holds that (a) \mathcal{F}_q is a C^∞ embedding, so we may identify $\mathcal{A}_{k-q} \times_{\mathcal{G}_{k-q}} Sp^q(E_{k-q})_{q-1}$ with its image $\mathcal{F}_q(\mathcal{A}_{k-q} \times_{\mathcal{G}_{k-q}} Sp^q(E_{k-q})_{q-1})$ into $\mathcal{IB}_{k,l}$ and (b) $\mathcal{A}_{k-q} \times_{\mathcal{G}_{k-q}} Sp^l(E_{k-q})_{q-1}$ is naturally embedded in $\mathcal{A}_{k-q} \times_{\mathcal{G}_{k-q}} Sp^q(E_{k-q})$.

For $1 \leq q \leq l$, we connect the spaces $\mathcal{JB}_{k,q-1}$ and $\mathcal{A}_{k-q} \times_{\mathcal{G}_{k-q}} Sp^q(E_{k-q})$ along the stratum $\mathcal{A}_{k-q} \times_{\mathcal{G}_{k-q}} Sp^q(E_{k-q})_{q-1}$, in order to define the space

^{†7} Recall that E_{k-l}^* is the bundle $P_{k-l} \times_{SU(2)} \mathbf{H} \xrightarrow{\pi} X_0$ with its zero section removed.

$$\mathcal{JB}_{k,l} = \mathcal{B}_k \coprod_{\mathcal{F}_1^*} \mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1} \coprod_{\mathcal{F}_2^*} \mathcal{A}_{k-2} \times_{\mathcal{G}_{k-2}} Sp^2(E_{k-2}) \coprod_{\mathcal{F}_3^*} \cdots \coprod_{\mathcal{F}_l^*} \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^l(E_{k-l}),$$

where $\mathcal{B}_k \coprod_{\mathcal{F}_1^*} \mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1} \equiv \mathcal{B}_k \coprod_{\mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1}^*} \mathcal{A}_{k-1} \times_{\mathcal{G}_{k-1}} E_{k-1}$, whilst we inductively define

$$\begin{aligned} & \mathcal{A}_{k-q} \times_{\mathcal{G}_{k-q}} Sp^q(E_{k-q}) \coprod_{\mathcal{F}_{q+1}^*} \mathcal{A}_{k-(q+1)} \times_{\mathcal{G}_{k-(q+1)}} Sp^{q+1}(E_{k-(q+1)}) \equiv \\ & \mathcal{A}_{k-q} \times_{\mathcal{G}_{k-q}} Sp^q(E_{k-q}) \coprod_{\mathcal{A}_{k-(q+1)} \times_{\mathcal{G}_{k-(q+1)}} Sp^{q+1}(E_{k-(q+1)})_q} \mathcal{A}_{k-(q+1)} \times_{\mathcal{G}_{k-(q+1)}} Sp^{q+1}(E_{k-(q+1)}). \end{aligned}$$

2.6 The Palais-Smale ends of \mathcal{B}_k

We combine the results of Section 2.4 with the construction of Section 2.5, to prove the existence of a ‘limit space’ for C -sequences of the moduli space \mathcal{B}_k . More precisely, we prove Theorem 0.2.1 of the Introduction, which is formally stated as follows.

Theorem 2.6.1 *Let $\pi : P_k \rightarrow X$ be an $SU(2)$ bundle and consider \mathcal{JB}_k to be the space constructed in Section 2.5 with its natural identification topology. Then, every C -sequence in \mathcal{B}_k has a convergent subsequence with limit point in \mathcal{JB}_k .*

Proof: We pick a C -sequence $\{[A_a]\}$ of orbits in \mathcal{B}_k . We recall that by definition, we have that $|\mathcal{YM}(A_a) - E| \rightarrow 0$ and $\|\nabla \mathcal{YM}_{A_a}\| \rightarrow 0$. According to Proposition 2.4.3, given $t \in (0, 1/8]$ and $\epsilon > 0$, there exists a subsequence of $\{[A_a]\}$, relabelled as $\{[A_a]\}$, and an index $a(\epsilon) < \infty$ such that for each $a > a(\epsilon)$ there exists a connection A on P_k with gauge orbit $[A] = \mathcal{R}_l(e[a])$, $0 \leq l \leq k$, and a limit sequence $e[a]$ of gluing data with $\|[A_a] - \mathcal{R}_l(e[a])\|_{L_1^2} < \epsilon$. We recall that \mathcal{R}_l is the iterated gluing map of §2.3.2 and $\{e[a]\}$ as in Definition 2.4.2. Consequently, the sequences $\{[A_a]\}$ and $\mathcal{R}_l(e[a])$ have identical limits.

Therefore, it is the limiting behaviour of $\mathcal{R}_l(e[a])$ which will determine the behaviour of the C -sequence $\{A_a\}$. As outlined in Section 2.3, the sequence $\mathcal{R}_l(e[a])$ is obtained by gluing a certain number of (glued) instantons over S^4 to a background connection A_0 on an $SU(2)$ -bundle $P_{k-l} \rightarrow X$ using the gluing data $e[a] = (A_0, p_1(a), \dots, p_l(a))$, where $\{p_1(a), \dots, p_l(a)\} \subset Sp^l(E_{k-l})$ is an unordered l -tuple of distinct points on X_0 .

We recall that due to the independence of the sequence $\{A_0(a)\}$ of background connections of the index a , it suffices to make a single choice for the background

connection.^{†8} We also choose the neck-width parameters λ_{p_i} , $1 \leq i \leq l$, of the gluing construction as in §1.5.1.

Let us note that the gluing map \mathcal{R}_l is a C^∞ embedding: on the level of connections, one checks fairly easy that the map is an embedding. To pass to the quotient, we patch together slice neighbourhoods in the component quotient spaces being glued up. The projection from small enough Coulomb-gauge neighbourhoods give diffeomorphisms from slice neighbourhoods into the quotient spaces. When we patch together these Coulomb-gauge slices, we obtain families of connections on the glued-up bundles which are ‘almost’ in Coulomb-gauge due to slight errors that the cut-off functions introduce. One then modifies the standard proof of the slice theorem [15] to show that the induced map on the level of quotients is an embedding.

On the other hand, by Corollary 1.6.14, the gluing map \mathcal{F}_l of Section 2.5 is also an embedding. Therefore, by construction the images of $\mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} D^l(E_{k-l})$ (see Section 0.1) in \mathcal{B}_k through \mathcal{R}_l and \mathcal{F}_l are diffeomorphic spaces. In spite of that the sequence $\mathcal{R}_l(e[a])$ lies in \mathcal{B}_k , its limit with respect to the topology induced by the L^2_1 -Sobolev norm does not. More precisely, we note that

$$\begin{aligned} \lim_{a \rightarrow \infty} \mathcal{R}_l &\equiv \lim_{a \rightarrow \infty} \mathcal{R}_l(A_0, \{p_1(a), p_2(a), \dots, p_l(a)\}) \\ &= \mathcal{F}_l(A_0, \{\lim_{a \rightarrow \infty} p_1(a), \lim_{a \rightarrow \infty} p_2(a), \dots, \lim_{a \rightarrow \infty} p_l(a)\}) . \end{aligned}$$

We observe that the terms of each sequence $\{p_i(a)\}$, $1 \leq i \leq l$, lie on the disc-bundle of the quaternionic bundle E_{k-l} since the construction requires $\{\lambda_{p_i}(a)\} \leq 1$. Hence, there is a subsequence $\{p_i(a)\}$ which converges to a limit and let (q_1, q_2, \dots, q_l) denote the limits of the sequences $\{p_i(a)\}$, $1 \leq i \leq l$. Thus, we have that $\lim_{a \rightarrow \infty} \{\mathcal{R}_l(e[a])\} = \mathcal{F}_l(A_0, \{q_1, q_2, \dots, q_l\})$.

On the other hand, Condition (c) of Definition 2.4.2 requires that the sequences of scales $\{\lambda_{p_i}(a)\}$, $1 \leq i \leq l$, are monotonically decreasing to zero. This asserts that the sequence $\{e[a]\} \in \mathcal{A}_{k-l} \times_{\mathcal{G}_{k-l}} Sp^{k-l}(E_{k-l})$ converges to $(A_0, \{x_1, x_2, \dots, x_l\})$ in the given topology, where $x_i = \pi(q_i)$, $1 \leq i \leq l$. Therefore, with respect to the natural identification topology, the C -sequence $\{[A_a]\}$ converges to $\mathcal{F}_l(A_0, \{x_1, x_2, \dots, x_l\})$ in $\mathcal{JB}_k = \coprod_{q=1}^k \coprod_{\mathcal{F}_q^{-1}} \mathcal{A}_{k-q} \times_{\mathcal{G}_{k-q}} Sp^q(E_{k-q})$. Hence, every C -sequence in \mathcal{B}_k converges with a limit point in the space \mathcal{JB}_k . \square

^{8†} See (a) of Definition 2.4.2.

Chapter 3

The μ -map in Gauge Theory

3.1 Introduction

Let X be a simply-connected, closed, oriented, Riemannian, 4-dimensional manifold and $P_k \rightarrow X$ be a principal $SU(2)$ -bundle of 2nd Chern class $c_2(P_k) = k$. Let \mathcal{M}_k and $\mathcal{IM}_k = \coprod_{q=0}^k \mathcal{M}_{k-q} \times Sp^q(X)$ denote the moduli spaces of irreducible ASD connections and ideal irreducible ASD connections on P_k , respectively. One of the most important ingredients in Donaldson theory is a map $\mu : H_2(X; \mathbf{Z}) \rightarrow H^2(\mathcal{M}_k; \mathbf{Z})$, which among other things provides the 2-dimensional generators of the polynomial algebra-structured rational cohomology of the moduli space \mathcal{B}_k^* of irreducible connections on P_k and also leads to the definition of the Donaldson polynomials which are differential-topological invariants of X . The construction of the map μ is presented in Section 3.2.

In Section 3.3, we follow §7.1.4 of [8] to study the construction of determinant line bundles over the compactified moduli space $\bar{\mathcal{M}}_k$ associated to certain families of Dirac operators over X , in order to show that the map μ actually descends to a homomorphism $\bar{\mu} : H_2(X) \rightarrow H^2(\bar{\mathcal{M}}_k)$. Moreover, we show that the above extension is such that for $z \in H^2(X)$, the restriction of the cohomology class $\bar{\mu}(z)$ to the l -th link $\bar{\mathcal{M}}_k \cap (\mathcal{M}_{k-l} \times Sp^l(X))$ yields the corresponding class $\mu^{(k-l)}(z) \in H^2(\mathcal{M}_{k-l})$ and the ‘symmetric sum’ $sp^l(PD(z))$ of the Poincare dual of z . This fact will be used extensively in Chapter 4 in a computation related to Donaldson polynomials.

3.2 The construction of the μ -map

Let \mathcal{B}_k^* be the infinite-dimensional moduli space of all connections on $P_k \rightarrow X$ and \mathcal{M}_k be the moduli space of equivalence classes of irreducible *ASD* connections on P_k is a submanifold of \mathcal{B}_k^* . Although the space \mathcal{A}_k of irreducible connections on P_k is topologically trivial, this is far from being true for the orbit space \mathcal{B}_k^* . Therefore, a question that immediately poses itself is the computation of the (co)homology of \mathcal{B}_k^* .

Another question concerning the topology of the underlying 4-manifold X is the construction of differential-topological invariants, i.e. functions which detect distinct smooth structures on an infinite family of equivalent homeomorphic structures for X .

The key ingredient that Donaldson theory uses to answer the above questions, is a homomorphism $\mu : H_2(X; \mathbf{Z}) \rightarrow H^2(\mathcal{B}_k^*; \mathbf{Z})$ which is defined as follows. Let \mathcal{G}_k^0 be a subgroup of the group of gauge transformations defined by $\mathcal{G}_k^0 = \{g \in \mathcal{G}_k : g_{x_0} = Id, x_0 \in X\}$.

Since P_k is a principal $SU(2)$ -bundle over X and the action of \mathcal{G}_k^0 on \mathcal{A}_k is free, the quotient $\mathcal{P}_k = \mathcal{A}_k \times_{\mathcal{G}_k^0} P_k$ defines a principal $SU(2)$ -bundle over $\mathcal{A}_k \times_{\mathcal{G}_k^0} X = \mathcal{B}_k^0 \times X$, where \mathcal{B}_k^0 is the moduli space of *framed* connections on P_k . Being an $SU(2)$ -bundle, \mathcal{P}_k is classified up to isomorphism by its 2nd Chern class $c_2(\mathcal{P}_k) \in H^4(\mathcal{B}_k^0 \times X)$.

The standard hypotheses on X imply that $H_0(X) \cong \mathbf{Z}, H_1(X) \cong 0, H_3(X) \cong 0, H_4(X) \cong \mathbf{Z}$. We apply Künneth's formula to decompose $H^4(\mathcal{B}_k^0 \times X)$, namely

$$H^4(\mathcal{B}_k^0 \times X) \cong H^4(\mathcal{B}_k^0) \otimes H^0(X) \oplus H^3(\mathcal{B}_k^0) \otimes H^1(X) \oplus H^2(\mathcal{B}_k^0) \otimes H^2(X) \oplus$$

$$H^1(\mathcal{B}_k^0) \otimes H^3(X) \oplus H^0(\mathcal{B}_k^0) \otimes H^4(X) \cong H^4(\mathcal{B}_k^0) \otimes \mathbf{Z} \oplus H^2(\mathcal{B}_k^0) \otimes H^2(X) \oplus H^0(\mathcal{B}_k^0) \otimes \mathbf{Z}.$$

With respect to this decomposition, let us write $c_2(\mathcal{P}_k) = c^{4,0} + c^{2,2} + c^{0,4}$. We now consider the *slant product* $| : H^2(\mathcal{B}_k^0) \otimes H^2(X) \otimes H_2(X) \rightarrow H^2(\mathcal{B}_k^0)$ defined by $|((a \times b) \otimes z) \equiv (a \otimes b)|z = \langle b, z \rangle \cdot a$, where $\langle \cdot, \cdot \rangle$ denotes the Kronecker pairing. We define a homomorphism $\mu_0 : H_2(X; \mathbf{Z}) \rightarrow H^2(\mathcal{B}_k^0; \mathbf{Z})$ by setting $\mu_0(z) = c^{2,2}|[z]$.

A detailed reasoning of the fact that the map μ_0 descends to a homomorphism $\mu : H_2(X; \mathbf{Z}) \rightarrow H^2(\mathcal{B}_k^*; \mathbf{Z})$ can be found in Section 5.2.1 of [8]. The key point is the use of the fact that \mathcal{B}_k^0 is a principal $SO(3)$ -bundle over \mathcal{B}_k^* (called *base point fibration*). Then, one constructs the *determinant line bundle* of the universal family of *Dirac operators* on the surface Σ representing $z \in H_2(X; \mathbf{Z})$ in order to obtain a line

bundle over \mathcal{B}_k^0 which represents $\mu_0(z)$. The significance of the construction stems from the fact that it also produces a representative for $\mu(z) \in H^2(\mathcal{B}_k^*; \mathbf{Z})$.

Let us remark here that the obstruction in lifting the classes $\mu(z)$ to the moduli space \mathcal{B}_k of all connections is due to the non-triviality of these classes on the links of reducible connections, which make up the complement $\mathcal{B}_k \setminus \mathcal{B}_k^*$ - see 5.1.4 of [8].

We now consider the composition $H_2(X) \xrightarrow{\mu} H^2(\mathcal{B}_k^*) \xrightarrow{i^*} H^2(\mathcal{M}_k)$, where the homomorphism i^* is induced by the natural inclusion i of \mathcal{M}_k in \mathcal{B}_k . One can show that the classes $\mu(z) \in H^2(\mathcal{M}_k; \mathbf{Z})$ extend to the compactified moduli space $\bar{\mathcal{M}}_k$. The argument involved is analytically explained in [8] and [17] and is outlined in Section 3.3. Hence, we finally obtain a homomorphism $\bar{\mu} : H_2(X) \rightarrow H^2(\bar{\mathcal{M}}_k)$.

The geometric interpretation of the map μ is examined in Chapter 5 of [8]. In fact, there are three ways by which a cohomology class $\mu(z) \in H^2(\bar{\mathcal{M}}_k)$ can be represented: as the 1st Chern class of a line bundle, as the Poincare dual of a codimension 2 submanifold of $\bar{\mathcal{M}}_k$, or as a class represented by a closed 2-form. For the relevant constructions, we refer to Sections 5.2-5.3 of [8].

Coming back to the use of the map μ : it provides the 2-dimensional generators of the rational cohomology of \mathcal{B}_k^* . More precisely, if z_1, \dots, z_b is a basis for $H_2(X)$, then, the rational cohomology of \mathcal{B}_k^* is a polynomial algebra on a 4-dimensional generator ν and the 2-dimensional generators $\mu(z_1), \dots, \mu(z_b)$. As far as the differential topology of X is concerned, one forms pairs of certain products of classes $\bar{\mu}(z)$ with the fundamental homology class of $\bar{\mathcal{M}}_k$ in order to produce the Donaldson polynomial invariants - see Chapter 10 of [8].

3.3 Line bundles over compactified moduli spaces

In this section, we outline a method of constructing a line bundle $\bar{\mathcal{L}}$ over the compactified moduli space $\bar{\mathcal{M}}_k$ of ASD connections, the restriction of which to the link $\bar{\mathcal{M}}_k \cap (\mathcal{M}_{k-l} \times Sp^l(X))$ of the l -th stratum $\mathcal{M}_{k-l} \times Sp^l(X)$ in $\bar{\mathcal{M}}_k$ gives rise to the cohomology class $\mu^{(k-l)}(z) \oplus sp^l(PD(z))$, $z \in H^2(X; \mathbf{Z})$, $l = 0, \dots, k$, where $sp^l(PD(z))$ denotes the l -th symmetric product of the Poincare dual of z .

The construction involves the extension of determinant line bundles over $\bar{\mathcal{M}}_k$, through the asymptotic analysis of the coupled Dirac operators with respect to the

‘distance’ to the boundary points of $\bar{\mathcal{M}}_k$. A detailed description of the construction can be found in §7.1.3-§7.1.5 of [8]. As we shall see in Chapter 4, this result provides the key to the derivation of formulae for Donaldson polynomials of X .

We first construct a line bundle \mathcal{L}_{k-j} over the moduli space \mathcal{M}_{k-j} , $0 \leq j \leq k$, which represents the cohomology classes $\mu^{(k-j)}(z)$, $z \in H^2(X)$ defined in Section 3.2. For simplicity, let us assume that X is a spin manifold and that z is divisible by two in the homology group, so there is a line bundle L over X such that $c_1(L^2) = PD(z)$, where $PD(z)$ denotes the Poincare dual of z .[†] We fix a connection ω on L and for any connection A on a bundle E over X , let $A + \omega$ be the induced connection on $E \otimes L$ and $A - \omega$ the induced connection on $E \otimes L^{-1}$, where L^{-1} denotes the dual of the bundle L . Let $\Lambda(A + \omega)$ be the *determinant line*

$$\Lambda(A + \omega) = \Lambda^{max}(Ker D_{A+\omega})^* \otimes \Lambda^{max}(Ker D_{A+\omega}^*),$$

associated to the *coupled Dirac operator* on $E \otimes L$.[‡] Similarly, we define the determinant line $\Lambda(A - \omega)$ of the Dirac operator on $E \otimes L^{-1}$. Let \mathcal{L}_{k-j} be the line bundle over the moduli space \mathcal{B}_{k-j}^* of irreducible $SU(2)$ -connections of charge $k - j$, the fibers of which are the lines $\mathcal{L}_{k-j,A} = \Lambda(A + \omega) \otimes [\Lambda(A - \omega)]^*$. One checks the action of the isotropy groups of A on the fibers $\mathcal{L}_{k-j,A}$ to ensure that the above definition descends to the quotient space - see §5.2.1 of [8].

Proposition 3.3.1 *The determinant line bundle \mathcal{L}_{k-j} is such that: $c_1(\mathcal{L}_{k-j}) = \mu^{(k-j)}(z)$, $z \in H^2(X; \mathbb{Z})$, $j = 0, \dots, k$.*

Proof: For the detailed proof, we refer to §7.1.3 of [8]. We sketch the main idea: choose a surface $\Sigma \subset X$ that represents the 2-dimensional homology class z and a section s of L cutting out Σ . Regard this as giving a trivialization of L outside a tubular neighbourhood N of Σ . Use the homotopy invariance of the index to notice the independence of the topological type of the determinant line bundle \mathcal{L}_{k-j} of the connection ω on L and choose ω to be flat outside N and compatible with the trivialization of L .

For a connection A on a bundle E over X , the coupled operators $D_1 = D_{A-\omega}$ and $D_2 = D_{A+\omega}$ are isomorphic in $X \setminus N$. We define a family of operators P_t such that $Ker P_0 = Ker D_1 \oplus Ker D_2^*$, $Ker P_0^* = Ker D_1^* \oplus Ker D_2$ and $P_1 = Id$ over $X \setminus N$.

[†] Note that both assumptions can be removed.

[‡] We refer to §5.1.3, §5.2.1 of [8].

We obtain a non-canonical isomorphism between \mathcal{L}_{k-j} and the line bundle $\det \text{ind} P_1$ formed from operators constructed canonically from A and with kernels supported in N . One copies §5.2.1 of [8] to check that the bundle \mathcal{L}_{k-j} has the desired 1st Chern class. \square

Next, we use the above description to show that \mathcal{L}_k actually extends over the compactified moduli space $\bar{\mathcal{M}}_k$. We define a quotient line bundle $Sp^l(L)$ over the l -th symmetric product $Sp^l(X)$ of X , by setting

$$Sp^l(L) = \frac{\pi_1^*(L) \otimes \dots \otimes \pi_l^*(L)}{S_l},$$

where $\pi_i : X^l \rightarrow X$ is the projection on the i -th factor, $1 \leq i \leq l$, and S_l is the permutation group of order l . Here, one uses the natural lift of the action of S_l to $\pi_1^*(L) \otimes \dots \otimes \pi_l^*(L)$ and the fact that the isotropy groups of points in the product act trivially on the fibers.

Theorem 3.3.2 *There is a line bundle $\bar{\mathcal{L}}$ over the compactified moduli space $\bar{\mathcal{M}}_k$ the restriction of which to the l -th link $\bar{\mathcal{M}}_k \cap (\mathcal{M}_{k-l} \times Sp^l(X))$ is isomorphic to the bundle $\mathcal{L}_{k-l} \otimes Sp^l(L)^2$.*

Proof: For the details of the proof, we refer to §7.1.4 – §7.1.5 of [8]. We survey the idea: choose an ideal ASD connection $(A, x_1, \dots, x_l) \in \mathcal{M}_{k-l} \times Sp^l(X)$ and consider a point $[A'] \in \mathcal{M}_k$ close to (A, x_1, \dots, x_l) . Since $[A']$ is close to A outside small balls, we can deform it to be equal to A outside these balls. An application of the excision construction gives isomorphisms

$$\Lambda(A' + \omega) \cong \Lambda(A + \omega) + \Lambda_+, \quad \Lambda(A' - \omega) \cong \Lambda(A - \omega) + \Lambda_-,$$

where the determinant lines Λ_+ , Λ_- are constructed from the indices of pseudo-differential operators P_+ , P_- which equal the identity outside small balls $B_r(x_i)$. We extend the above isomorphisms to the intersection of \mathcal{M}_k with a neighbourhood N in $\bar{\mathcal{M}}_k$ of (A, x_1, \dots, x_l) and proceed to choose P_+ , P_- such that \mathcal{L}_k is trivial over $\mathcal{M}_k \cap N$.

However, to extend \mathcal{L}_k over $\bar{\mathcal{M}}_k$ one needs to show the way that the lines $Sp^l(L)$ enter the picture: since the operators P_+ , P_- are equal to the identity outside the disjoint small balls $B_r(x_i)$, they can be viewed as collections of operators P_+^i , P_-^i with

corresponding determinant lines Λ_+^i, Λ_-^i , where now the tuple (x_1, \dots, x_l) is written as a collection of points x_i with multiplicities m_i , $1 \leq i \leq l$.

We then define natural isomorphisms $\text{Ker } P_+^i \cong (\text{Ker } P_-^i) \otimes L_{x_i}^2$. An application of standard theory gives then a natural isomorphism $\det \text{ind} P_+ - \det \text{ind} P_- = Sp_{(x_1, \dots, x_l)}^{l^2}$. In 7.1.4 of [8], the excision theory yields that there is a neighbourhood N of (A, x_1, \dots, x_l) in $\bar{\mathcal{M}}_k$ and a family of operators P_t parametrized by the stratum $N \cap (\mathcal{M}_{k-l} \times Sp^l(X))$ such that the determinant line bundle of P_0 is isomorphic to $\mathcal{L}_{k-l} \otimes Sp^l(L)^2$ and the determinant line bundle of P_1 is isomorphic to the line $\mathcal{L}_{k-l, A} \otimes L_{x_1}^2 \otimes \dots \otimes L_{x_l}^2$.

To complete the proof, one shows that the choice of isomorphism between the determinant lines of P_0 and P_1 can be restricted in a way that the associated local trivializations of $\bar{\mathcal{L}}$ are compatible on the overlap between the corresponding neighbourhoods in $\bar{\mathcal{M}}_k$. The geometric construction concerning the specification of such trivializations takes up §7.1.5 of [8]. \square

3.4 The extension of μ over $\bar{\mathcal{M}}_k$

In this section, we observe that the results obtained in Section 3.3 lead to the extension of the map $\mu : H_2(X) \rightarrow H^2(\mathcal{M}_k)$, described in Section 3.2, to the compactified moduli space $\bar{\mathcal{M}}_k$. Moreover, we show that the restriction of the extended map $\bar{\mu} : H_2(X) \rightarrow H^2(\bar{\mathcal{M}}_k)$ to the lower strata of the compactification yields the corresponding μ -class and symmetric sums of the Poincare-dual of a reference class $z \in H^2(X; \mathbb{Z})$. More precisely, let us apply Theorem 3.3.2 for the value $l = 1$ and observe that by construction the 1st Chern class of the bundle $\mathcal{L}_{k-1} \otimes L^2$ is

$$c_1(\mathcal{L}_{k-1} \otimes L^2) = c_1(\mathcal{L}_{k-1}) \otimes 1 + 1 \otimes c_1(L^2) \equiv c_1(\mathcal{L}_{k-1}) + c_1(L^2).$$

Let $PD(z) \in H^2(X; \mathbb{Z})$ be the Poincare-dual of $z \in H_2(X; \mathbb{Z})$ and also $\mu^{(k-1)}(z) \in H^2(\mathcal{M}_{k-1})$ be the associated μ -class for an $SU(2)$ -bundle P_{k-1} of class $c_2(P_{k-1}) = c_2(P_k) - 1 \equiv k - 1$. We use our hypothesis on z , namely $c_1(L^2) = PD(z)$, and apply Proposition 3.3.1 for $j = 1$, to obtain that

$$c_1(\mathcal{L}_{k-1} \otimes L^2) = \mu^{(k-1)}(z) \oplus PD(z) \in H^2(\mathcal{M}_{k-1} \times X).$$

Hence, the construction of Section 4.3 actually produces a cohomology class, namely the class $\bar{\mu}(z) = c_1(\bar{\mathcal{L}}) \in H^2(\bar{\mathcal{M}}_k)$, the restriction of which to the stratum $\bar{\mathcal{M}}_k \cap (\mathcal{M}_{k-1} \otimes X)$ yields the class

$$\pi_1^*(\mu^{(k-1)}(z)) + \pi_2^*(PD(z)) \in H^2(\mathcal{M}_{k-1} \times X) ,$$

where π_i is the projection of $\mathcal{M}_{k-1} \times X$ on the i -th factor, $i = 1, 2$. We recall that the standard hypotheses on X yield that $H_0(X) \cong \mathbf{Z}$, $H_1(X) \cong \mathbf{0}$, $H_3(X) \cong \mathbf{0}$, $H_4(X) \cong \mathbf{Z}$. We apply Künneth's formula to decompose $H^2(\mathcal{M}_{k-1} \times X)$, namely

$$\begin{aligned} H^2(\mathcal{M}_{k-1} \times X) &\cong H^2(\mathcal{M}_k) \otimes H^0(X) \oplus H^1(\mathcal{M}_{k-1}) \otimes H^1(X) \oplus H^0(\mathcal{M}_{k-1}) \otimes H^2(X) \\ &\cong H^2(\mathcal{M}_{k-1}) \otimes \mathbf{Z} \oplus \mathbf{Z} \otimes H^2(X) \cong H^2(\mathcal{M}_{k-1}) \oplus H^2(X). \end{aligned}$$

Thus, one obtains an isomorphism $\varphi : H^2(\mathcal{M}_{k-1} \times X) \longrightarrow H^2(\mathcal{M}_{k-1}) \oplus H^2(X)$ which identifies the class $\pi_1^*(\mu^{(k-1)}(z)) + \pi_2^*(PD(z))$ with $\mu^{(k-1)}(z) \oplus PD(z)$.

Let us now deal with the general case. For $z \in H_2(X; \mathbf{Z})$, let $sp^l(PD(z))$ be the l -th 'symmetric sum' of copies of $PD(z)$ and $\mu^{(k-l)}(z) \in H^2(\mathcal{M}_{k-l})$ be the μ -class associated to a reference $SU(2)$ -bundle P_{k-l} of class $c_2(P_k) - l \equiv k - l$. Let us apply Theorem 3.3.2 in the case $l \geq 2$. We note that the 1st Chern class of the bundle $\mathcal{L}_{k-l} \otimes Sp^l(L^2)$ is

$$c_1(\mathcal{L}_{k-l} \otimes Sp^l(L^2)) = c_1(\mathcal{L}_{k-l}) \otimes 1 + 1 \otimes c_1(Sp^l(L^2)) \equiv c_1(\mathcal{L}_{k-l}) + c_1(Sp^l(L^2)).$$

We use our hypothesis on z and apply Proposition 3.3.1 to obtain that

$$c_1(\mathcal{L}_{k-l} \otimes Sp^l(L^2)) = \pi_1^*(\mu^{(k-l)}(z)) + \pi_2^*(Sp^l(PD(z))) \in H^2(\mathcal{M}_{k-l} \times Sp^l(X)) ,$$

where π_i is the projection of $\mathcal{M}_{k-l} \times Sp^l(X)$ on the i -th factor, $i = 1, 2$.^{3†} Hence, the restriction of the line bundle $\bar{\mathcal{L}}$ over $\bar{\mathcal{M}}_k$ to the stratum $\bar{\mathcal{M}}_k \cap (\mathcal{M}_{k-l} \times Sp^l(X))$ is isomorphic to $\mathcal{L}_{k-l} \otimes Sp^l(L^2)$. Equivalently, its 1st Chern class $c_1(\bar{\mathcal{L}}) = \bar{\mu}(z) \in H^2(\bar{\mathcal{M}}_k)$ restricts to $\bar{\mathcal{M}}_k \cap (\mathcal{M}_{k-1} \times X)$ to the class $\pi_1^*(\mu^{(k-l)}(z)) + \pi_2^*(Sp^l(PD(z)))$.

The standard hypotheses on X and the application of Künneth's formula yield the following decomposition for $H^2(\mathcal{M}_{k-l} \times Sp^l(X))$:

$$H^2(\mathcal{M}_{k-l} \times Sp^l(X)) \cong H^2(\mathcal{M}_{k-l}) \otimes H^0(Sp^l(X)) \oplus H^1(\mathcal{M}_{k-l}) \otimes H^1(Sp^l(X)) \oplus$$

^{3†} See also Section 2(a) of [11].

$$H^0(\mathcal{M}_{k-l}) \otimes H^2(Sp^l(X)) \cong H^2(\mathcal{M}_{k-l}) \otimes \mathbb{Z} \oplus \mathbb{Z} \otimes H^2(Sp^l(X)) \cong H^2(\mathcal{M}_{k-l}) \oplus H^2(Sp^l(X)).$$

Hence, we obtain an isomorphism $\varphi_l : H^2(\mathcal{M}_{k-l} \times Sp^l(X)) \longrightarrow H^2(\mathcal{M}_{k-l}) \oplus H^2(Sp^l(X))$ which identifies the classes $\pi_1^*(\mu^{(k-l)}(z)) + \pi_2^*(Sp^l(PD(z)))$ and $\mu^{(k-l)}(z) \oplus Sp^l(PD(z))$.

Remark 3.4.1 To extend the map μ of Section 3.2 over the compactified moduli space $\bar{\mathcal{M}}_k$, one simply employs the results of Sections 3.3, 3.4 to define a homomorphism $\bar{\mu} : H_2(X) \longrightarrow H^2(\bar{\mathcal{M}}_k)$ by setting $\bar{\mu}(z) = c_1(\bar{\mathcal{L}})$, $z \in H^2(X)$.

In fact, the procedure of Section 3.3 produces a class $\tilde{\mu}(z) \in H^2(\mathcal{IM}_k)$ whose restriction to $H^2(\mathcal{M}_{k-l} \times Sp^l(X))$ yields the class $\mu^{(k-l)}(z) \oplus sp^l(PD(z))$. The fact that the class $\tilde{\mu}(z)$ is a natural extension of the class $\bar{\mu}(z) \in H^2(\bar{\mathcal{M}}_k)$ follows explicitly from the proof of Theorem 3.3.2 since the argument used there applies unchanged when we replace $\bar{\mathcal{M}}_k$ by \mathcal{IM}_k .

Alternatively, one can employ standard algebraic-topological techniques to show that the moduli spaces $\bar{\mathcal{M}}_k$ and \mathcal{IM}_k have isomorphic 2-dimensional cohomology groups since they differ by a set of codimension greater than or equal to 4.

Chapter 4

Universal Formulae for Donaldson Invariants

4.1 Main Results

One of the main aims of 4-dimensional geometry has been the classification of smooth 4-manifolds up to diffeomorphism. Despite that the problem is still widely open, some of the sophisticated theories developed to provide an answer have given partial results of major importance. More precisely, employing the moduli spaces of solutions of the anti-self-dual Yang-Mills equations in 4 dimensions, in 1990, S. Donaldson [9] discovered new invariants for smooth, closed, oriented, Riemannian 4-manifolds.

The crucial observation was that despite that the Yang-Mills equations depend on the Riemannian geometry of the underlying 4-manifold X , at the level of homology one finds properties of the moduli spaces which are invariant under the variation of the metric.

The Donaldson polynomials detect distinct differentiable structures within an infinite family of homeomorphically equivalent structures. Since their computation even in relatively simple cases has been quite enigmatic, the effort to understand their structure has been often focused on establishing constraints or universal relations which they satisfy. A characteristic sample is provided by the works of Y. Ruan [36], P. Kronheimer and T. Mrowka [28], R. Fintushel and R. Stern [18].

In Section 4.2, we give the definition of Donaldson polynomials and also state their

main properties. For completeness, we state theorems which describe the behaviour of the polynomials for the cases of decomposable 4-manifolds and algebraic surfaces.

In Section 4.3, we use the natural inclusion of the compactified moduli space $\bar{\mathcal{M}}_k$ of *ASD* connections into the ideal *ASD* moduli space \mathcal{IM}_k and employ certain properties of the ‘ideal’ homomorphism $\tilde{\mu} : H_2(X; \mathbf{Z}) \longrightarrow H^2(\mathcal{IM}_k, \mathbf{Z})$ described in Chapter 3, in order to give a modified description of the Donaldson polynomials, using standard algebraic-topological arguments.

More precisely, under the assumption that the moduli space \mathcal{M}_k of *ASD* connections of charge k is even-dimensional, $2d$ say, where $d = 4k - \frac{3}{2}(1 + b^+)$, we show that for k within the *stable range* and for a natural homology class $[\mathcal{IM}_k] \in H^{2d}(\mathcal{IM}_k; \mathbf{Z})$, the polynomial functions $\mathcal{D}_k : H^2(X; \mathbf{Z}) \times \dots \times H^2(X; \mathbf{Z}) \longrightarrow \mathbf{Z}$ defined by the Kronecker pairing $\mathcal{D}_k(z_1, \dots, z_d) = \langle \tilde{\mu}(z_1) \dots \tilde{\mu}(z_d), [\mathcal{IM}_k] \rangle$ provide an ‘alternative’ viewing of the Donaldson polynomials.

This modified description of the invariants stems its importance from the fact that one can now evaluate certain products of $\tilde{\mu}$ -classes to any appropriate - with respect to the dimension - stratum of the moduli space \mathcal{IM}_k . It is the computation of those pairings which yields families of universal formulae which among other functions also include Donaldson polynomials of certain degree. The motivation for performing this ‘low-dimensional weighting’ of the classes $\tilde{\mu}(z_i)$ arises from the observation that the evaluation of the cohomology of \mathcal{IM}_k on its top-dimensional homology overlooks a great ammount of topological information that the space carries.

More precisely, let $j_l : \bar{\mathcal{M}}_{k-l} \times Sp^l(X) \longrightarrow \mathcal{IM}_k$ denote the natural inclusion: $\bar{\mathcal{M}}_{k-l} \times Sp^l(X) \hookrightarrow \mathcal{IM}_{k-l} \times Sp^l(X) \hookrightarrow \mathcal{IM}_k$, $0 \leq l \leq k$. We consider the restriction of the cohomology class $\tilde{\mu}(z_1) \dots \tilde{\mu}(z_{d-2l}) \in H^{2d-4l}(\mathcal{IM}_k)$ to proper boundary components of \mathcal{IM}_k . Namely, let \mathcal{F}_d^l denote the pairing

$$\mathcal{F}_d^l(z_1, \dots, z_{d-2l}) \equiv \langle (j_l)^*(\tilde{\mu}(z_1) \dots \tilde{\mu}(z_{d-2l})), [\bar{\mathcal{M}}_{k-l} \times Sp^l(X)] \rangle, \quad 0 \leq l \leq k.$$

Theorem 4.1.1 *Let P_k be an $SU(2)$ -bundle over X and D_{k-l} be a Donaldson polynomial of degree $2d - 4l$, $0 \leq l \leq k$. Let $\bar{z} \equiv PD(z)$ denote the Poincare-dual of $z \in H^2(X; \mathbf{Z})$ and S_{d-2l} be the permutation group of order $d - 2l$. Let also $Q^{(l)}$ be a multilinear form defined by the expression*

$$Q^{(l)}(z_1, \dots, z_{2l}) = \frac{1}{2^l l!} \sum_{\sigma \in S_{2l}} Q(z_{\sigma(1)}, z_{\sigma(2)}) \cdot \dots \cdot Q(z_{\sigma(2l-1)}, z_{\sigma(2l)}),$$

where Q denotes the intersection form of X . For $z_i \in H^2(X; \mathbf{Z})$, $1 \leq i \leq d - 2l$, it holds that

$$\mathcal{F}_d^l(z_1, \dots, z_{d-2l}) = \begin{cases} \sum_{\sigma \in S_{d-2l}} D_{k-l}(z_{\sigma(1)}, \dots, z_{\sigma(d-4l)}) \cdot Q^{(l)}(\bar{z}_{\sigma(d-4l+1)}, \dots, \bar{z}_{\sigma(d-2l)}), & \text{if } d > 4l \\ Q^{(l)}(\bar{z}_1, \dots, \bar{z}_{2l}), & \text{if } d = 4l. \end{cases}$$

4.2 Donaldson Polynomials: an overview

This section surveys the definition of the Donaldson polynomials and their main properties. For a detailed description, we refer to [11], [12] and Chapter 9 of [8].

Let us recall that the intersection form of a simply-connected, closed, oriented, Riemannian 4-manifold X is a symmetric integer matrix of determinant ± 1 , defined by the intersection properties of the 2-cycles of X and which depends on a choice of basis for $H_2(X; \mathbf{Z})$. For the theory of symmetric bilinear forms, we refer to [34]. A conclusion of Donaldson's theory is that if the intersection form Q_X of a smooth 4-manifold X is definite, then Q_X is equivalent to the standard diagonal form over the integers.^{†1}

Apparently, Donaldson's theorem implies the existence of an infinite number of smooth 4-manifolds with the same intersection form. These smooth 4-manifolds are in the same homeomorphic class, since Freedman's theorem [16] proves that the intersection form is a complete invariant for the homeomorphic type of smooth 4-manifolds. The Donaldson polynomial invariants were introduced [9] to distinguish the members of that infinite family which are not diffeomorphic.

The main idea concerning the definition of the invariants is the formation of a Kronecker pairing between a 'distinguished' cohomology class of the moduli space \mathcal{M}_k of ASD connections, which is independent of the choice of the Riemannian metric used to define \mathcal{M}_k , and a 'fundamental' class of \mathcal{M}_k . However, the obstacle of this scheme is the probable non-compactness of \mathcal{M}_k which prevents us from defining rigorously a fundamental class $[\mathcal{M}_k]$. The problem can be overcome using the Uhlenbeck compactification of the moduli space \mathcal{M}_k described in Section 1.4. On the other hand, the desired 2-dimensional homology classes are provided by the extension of the μ -map of Section 3.2 to the compactified moduli space.

^{†1} See Chapter 8 of [8] or [10].

Definition 4.2.1 Let X be a simply-connected 4-manifold with $b^+(X) = 2a + 1$, $a \geq 1$, where $b^+(X)$ is the number of positive entries of the intersection matrix $[Q]$ after its diagonalization over the real numbers. Let us assume that the dimension of the moduli space \mathcal{M}_k of ASD connections is $2d$, where $d = 4k - \frac{3}{2}(1 + b^+)$.^{†2} Let $k > \frac{1}{4}(3b^+ + 5)$ and also $\bar{\mu}$ be the map of Chapter 3. The *Donaldson polynomial* of charge k is a map $D_k : H^2(X; \mathbf{Z}) \times \dots \times H^2(X; \mathbf{Z}) \longrightarrow \mathbf{Z}$ defined by the pairing

$$D_k(z_1, \dots, z_d) = \langle \bar{\mu}(z_1) \dots \bar{\mu}(z_d), [\bar{\mathcal{M}}_k] \rangle, \quad z_i \in H^2(X; \mathbf{Z}), \quad 1 \leq i \leq d,$$

where $[\bar{\mathcal{M}}_k] \in H_{2d}(\bar{\mathcal{M}}_k; \mathbf{Z})$ denotes the fundamental class of the compactified moduli space $\bar{\mathcal{M}}_k$.

Remark 4.2.2 (a) The constraint $b^+(X) = 2a + 1$, $a \geq 1$, is imposed to ensure that d is an integer. In the case that $b^+ = 1$, one can still define differential-topological invariants for X which are of much more complicated form - see [26].

(b) The *stable range condition* $k > \frac{1}{4}(3b^+ + 5)$ or equivalently $d \geq 2k + 1$, is imposed to ensure that $\bar{\mathcal{M}}_k$ is generically a manifold except at a singular set of codimension at least 2. This follows from standard homology theory and is the usual condition for a singular complex to possess a fundamental class - see §9.2.1 of [8].

(c) As remarked in Section 3.3, one must verify that the classes $\mu(z_i) \in H^2(\mathcal{M}_k; \mathbf{Z})$, $1 \leq i \leq d$, extend to the compactified moduli space $\bar{\mathcal{M}}_k$. The main argument of this extension concerns the construction of determinant line bundles over $\bar{\mathcal{M}}_k$ as presented in §7.1.3 – §7.1.5 of [8].

Let us state the key properties of the Donaldson polynomials, the proofs of which take up Chapters 9 and 10 of [8].

- (a) The polynomial D_k depends on z_i only through its homology class $[z_i]$, $1 \leq i \leq d$.
- (b) The polynomial D_k is multilinear and symmetric in $[z_i]$, $1 \leq i \leq d$.
- (c) The polynomial D_k is independent of the choice of the Riemannian metric of X - see §2(a) of [11] - and it is an invariant of the oriented diffeomorphic type of X . More precisely, if Y is a smooth 4-manifold and $f : X \longrightarrow Y$ an orientation-preserving diffeomorphism, then $D_{k,X}(z_1, \dots, z_d) = D_{k,Y}(f_*(z_1), \dots, f_*(z_d))$.

^{2†} For the derivation of the dimension-formula for moduli spaces using the Atiyah-Singer *index theorem*, we refer to §4.2.5 of [8].

- (d) Let X be a simply-connected 4-manifold with $b^+(X) = 2a + 1$, $a \geq 1$, such that X is a connected sum $X = X_1 \# X_2$ with $b^+(X_i) > 0$, $i = 1, 2$. Then, for all k , the polynomials D_k are identically zero - see Section 9.3 of [8].
- (e) Let S be a complex algebraic surface with $b^+ \geq 3$ and odd. Then, for all k , the polynomials D_k vanish nowhere - see Chapter 10 of [8] or §2(c) of [11].

4.2.1 An ideal description of Donaldson Polynomials

We show that the construction of Section 3.4 provides us with a modified description of the original definition of the Donaldson polynomials.

As already discussed in Remark 4.2.2, despite that the ideal moduli space \mathcal{IM}_k of ASD connections is not a manifold, it can still possess a fundamental class on its top-dimensional homology provided that the codimension of its strata $M_{k-q} \times Sp^q(X)$, $0 \leq q \leq k$, is at least 2. This is equivalent to saying that the formal dimension, $2d$ say, of \mathcal{IM}_k must exceed the dimension of its lowest stratum $M_0 \times Sp^k(X)$ by at least 2. Since $\dim Sp^k(X) = 4k$, we therefore constraint d to obey the inequality $2d \equiv 8k - 3(1 + b^+) \geq 4k + 2$ or equivalently $k \geq \frac{1}{4}(3b^+ + 5)$.

For $d = 4k - \frac{3}{2}(1 + b^+)$, we choose classes z_1, \dots, z_d in $H^2(X; \mathbf{Z})$ and let $\tilde{\mu}(z_i)$, $1 \leq i \leq d$, be the corresponding cohomology classes described in Section 3.4. Under the assumptions of Definition 4.2.1, we define a polynomial function $\mathcal{D}_k : H^2(X; \mathbf{Z}) \times \dots \times H^2(X; \mathbf{Z}) \rightarrow \mathbf{Z}$ by the Kronecker pairing $\mathcal{D}_k(z_1, \dots, z_d) = \langle \tilde{\mu}(z_1) \dots \tilde{\mu}(z_d), [\mathcal{IM}_k] \rangle$, where $[\mathcal{IM}_k] \in H^{2d}(\mathcal{IM}_k; \mathbf{Z})$ denotes the fundamental class of \mathcal{IM}_k .

We now pick classes $z_1, \dots, z_d \in H^2(X; \mathbf{Z})$ and let $[\bar{\mathcal{M}}_k]$ and $[\mathcal{IM}_k]$ denote the fundamental classes of the moduli spaces $\bar{\mathcal{M}}_k$ and \mathcal{IM}_k , respectively. Let $i : \bar{\mathcal{M}}_k \rightarrow \mathcal{IM}_k$ be the natural inclusion and $i_* : H_{2d}(\bar{\mathcal{M}}_k) \rightarrow H_{2d}(\mathcal{IM}_k)$ be the induced map in homology.

We then have that $i_*([\bar{\mathcal{M}}_k]) = [\mathcal{IM}_k]$ (notice that the map i_* is of degree 1). We recall that by construction the restriction of the map $\tilde{\mu} : H_2(X; \mathbf{Z}) \rightarrow H^2(\mathcal{IM}_k, \mathbf{Z})$ to the compactification $\bar{\mathcal{M}}_k$ gives the map $\bar{\mu}$ of Section 3.4. Therefore, we obtain that

$$\mathcal{D}_k(z_1, \dots, z_d) = \langle \tilde{\mu}(z_1) \dots \tilde{\mu}(z_d), [\mathcal{IM}_k] \rangle = \langle \tilde{\mu}(z_1) \dots \tilde{\mu}(z_d), i_*[\bar{\mathcal{M}}_k] \rangle =$$

$$\langle i^*(\tilde{\mu}(z_1) \dots \tilde{\mu}(z_d)), [\bar{\mathcal{M}}_k] \rangle = \langle \bar{\mu}(z_1) \dots \bar{\mu}(z_d), [\bar{\mathcal{M}}_k] \rangle = D_k(z_1, \dots, z_d),$$

where D_k is a Donaldson polynomial of degree $2d$ and k is within the stable range.

4.3 Universal formulae for Donaldson Invariants

We use the modified description of the Donaldson polynomials described in §4.2.1, to derive a 2-parameter family of formulae which these polynomials satisfy. Let $\tilde{\mu} : H_2(X; \mathbf{Z}) \rightarrow H^2(\mathcal{IM}_k, \mathbf{Z})$ be as in Chapter 3 and $2d = 8k - 3(1 + b^+)$ be the dimension of the moduli space \mathcal{M}_k .

Let j_l denote the inclusion $\bar{\mathcal{M}}_{k-l} \times Sp^l(X) \hookrightarrow \mathcal{IM}_{k-l} \times Sp^l(X) \hookrightarrow \mathcal{IM}_k$, $0 \leq l \leq k$, and $j_l^* : H^{2d-4l}(\mathcal{IM}_k) \rightarrow H^{2d-4l}(\bar{\mathcal{M}}_{k-l} \times Sp^l(X))$ be the induced map in cohomology.

We aim to evaluate certain products of $\tilde{\mu}$ -cohomology classes on the intermediate stratum $\bar{\mathcal{M}}_{k-l} \times Sp^l(X)$. More precisely, for $z_i \in H_2(X; \mathbf{Z})$, $1 \leq i \leq d - 2l$, we will compute the function

$$\mathcal{F}_d^l(z_1, \dots, z_{d-2l}) = \langle (j_l)^*(\tilde{\mu}(z_1) \dots \tilde{\mu}(z_{d-2l})), [\bar{\mathcal{M}}_{k-l} \times Sp^l(X)] \rangle, \quad 0 \leq l \leq k.$$

Remark 4.3.1 Despite that the l -th symmetric product $Sp^l(X)$, $0 \leq l \leq k$, has singularities, it still carries a fundamental class $[Sp^l(X)] \in H_{4l}(Sp^l(X))$ since its diagonal Δ_l has codimension 4 - see also Remark 4.2.1.

4.3.1 A particular case

To get an indication about the form of \mathcal{F}_d^l , let us first carry out the computation for low values of l and d . More precisely, let us choose $l = 2$ and consider classes $z_i \in H_2(X; \mathbf{Z})$, $1 \leq i \leq d - 4$. Therefore, we want to compute the quantity

$$\mathcal{F}_d^2(z_1, \dots, z_{d-4}) = \langle (j_2)^*(\tilde{\mu}(z_1) \dots \tilde{\mu}(z_{d-4})), [\bar{\mathcal{M}}_{k-2} \times Sp^2(X)] \rangle.$$

Let us recall that the dimension formula gives that $\dim \bar{\mathcal{M}}_k = 2d = 8k - 3(1 + b^+)$. Since $\dim \bar{\mathcal{M}}_{k-2} = 2d - 16$, the moduli space $\bar{\mathcal{M}}_{k-2}$ is non-empty if $d \geq 8$.

(a). We choose $d = 8$. We need to determine the class $(j_2)^*(\tilde{\mu}(z_1) \dots \tilde{\mu}(z_4))$. A review of the construction and properties of the map $\tilde{\mu}$, will convince the reader that for the case in question it holds that

$$(j_2)^*(\tilde{\mu}(z_1) \dots \tilde{\mu}(z_4)) = [\mu(z_1) \otimes 1 + 1 \otimes b_1] \cdot [\mu(z_2) \otimes 1 + 1 \otimes b_2] \cdot [\mu(z_3) \otimes 1 + 1 \otimes b_3]$$

$$\cdot [\mu(z_4) \otimes 1 + 1 \otimes b_4],$$

where b_i is the ‘symmetric sum’ of copies of the Poincare-dual of the homology class z_i , $1 \leq i \leq 4$. Namely, let $\pi_j : X \times X \rightarrow X$, $j = 1, 2$, be the natural j -th projection and $\pi_j^* : H^2(X) \rightarrow H^2(X \times X)$ be the induced map in cohomology. For $z_i \in H^2(X; \mathbb{Z})$, let \bar{z}_i denote the Poincare-dual of z_i , $1 \leq i \leq 4$. We define the classes $b_i \in H^2(Sp^2(X))$ by $b_i = [\pi_1^*(\bar{z}_i) + \pi_2^*(\bar{z}_i)] \equiv [\bar{z}_i \otimes 1 + 1 \otimes \bar{z}_i]$, where the brackets $[]$ indicate the ‘symmetrization’ of $\pi_1^*(\bar{z}_i) + \pi_2^*(\bar{z}_i)$.^{†3} A straightforward calculation gives that

$$\begin{aligned} (j_2)^*(\tilde{\mu}(z_1), \tilde{\mu}(z_2), \tilde{\mu}(z_3), \tilde{\mu}(z_4)) &= \mu(z_1)\mu(z_2)\mu(z_3)\mu(z_4) \otimes 1 + [\mu(z_1)\mu(z_2)\mu(z_3) \otimes b_4 + \\ &\mu(z_1)\mu(z_2)\mu(z_4) \otimes b_3 + \mu(z_1)\mu(z_3)\mu(z_4) \otimes b_2 + \mu(z_2)\mu(z_3)\mu(z_4) \otimes b_1] + [\mu(z_1)\mu(z_2) \otimes b_3 b_4 + \\ &\mu(z_1)\mu(z_3) \otimes b_2 b_4 + \mu(z_1)\mu(z_4) \otimes b_2 b_3 + \mu(z_2)\mu(z_4) \otimes b_1 b_3 + \mu(z_3)\mu(z_4) \otimes b_1 b_2] + \\ &[\mu(z_1) \otimes b_2 b_3 b_4 + \mu(z_2) \otimes b_1 b_3 b_4 + \mu(z_3) \otimes b_1 b_2 b_4 + \mu(z_4) \otimes b_1 b_2 b_3] + 1 \otimes b_1 b_2 b_3 b_4 . \end{aligned}$$

Note that if $i_2 : Sp^2(X) \rightarrow \mathcal{IM}_k$ is the obvious inclusion, it holds that

$$\langle i_2^*(\tilde{\mu}(z_1)\tilde{\mu}(z_2)\tilde{\mu}(z_3)\tilde{\mu}(z_4)), [Sp^2(X)] \rangle = \langle b_1 b_2 b_3 b_4, [Sp^2(X)] \rangle = Q^{(2)}(\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4) ,$$

where the function $Q^{(2)}$ is defined in terms of the intersection form Q of the 4-manifold X by the formula

$$Q^{(2)}(\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4) = \frac{1}{2^2 2!} \sum_{\sigma \in S_4} Q(\bar{z}_{\sigma(1)}, \bar{z}_{\sigma(2)}) Q(\bar{z}_{\sigma(3)}, \bar{z}_{\sigma(4)}) .$$

Since $\dim \bar{\mathcal{M}}_{k-2} = 0$, a dimension-counting argument shows that the only term which participates in our computation is $1 \otimes b_1 b_2 b_3 b_4$. Therefore, we obtain that

$$\mathcal{F}_8^2(z_1, z_2, z_3, z_4) = Q^{(2)}(\bar{z}_1, \bar{z}_2, \bar{z}_3, \bar{z}_4) .$$

(b). We choose $d = 9$. We copy the argument of the previous case to obtain that

^{3†} We note that the class $\bar{z}_i \otimes 1 + 1 \otimes \bar{z}_i \in H^2(X \times X)$ is invariant under the action of the symmetric group S_2 . If $q : X \times X \rightarrow Sp^2(X)$ is the quotient map and $H^2(X \times X)^{S_2}$ consists of the S_2 -invariant 2-cohomology classes of $X \times X$, then, $q^* : H^2(Sp^2(X)) \rightarrow H^2(X \times X)^{S_2}$ is an isomorphism. By $b = [\bar{z}_i \otimes 1 + 1 \otimes \bar{z}_i] \in H^2(Sp^2(X))$ we denote the unique class which is such that $q^*(c) = \bar{z}_i \otimes 1 + 1 \otimes \bar{z}_i$.

$$\begin{aligned}
(j_2)^*(\tilde{\mu}(z_1), \tilde{\mu}(z_2), \tilde{\mu}(z_3), \tilde{\mu}(z_4), \tilde{\mu}(z_5)) &= \mu(z_1)\mu(z_2)\mu(z_3)\mu(z_4)\mu(z_5) \otimes 1 + \\
&[\mu(z_1)\mu(z_2)\mu(z_3)\mu(z_4) \otimes b_5 + \mu(z_1)\mu(z_2)\mu(z_3)\mu(z_5) \otimes b_4 + \mu(z_1)\mu(z_2)\mu(z_4)\mu(z_5) \otimes b_3 + \\
&\mu(z_1)\mu(z_3)\mu(z_4)\mu(z_5) \otimes b_2 + \mu(z_2)\mu(z_3)\mu(z_4)\mu(z_5) \otimes b_1] + [\mu(z_1)\mu(z_2)\mu(z_3) \otimes b_4b_5 + \\
&\mu(z_1)\mu(z_2)\mu(z_4) \otimes b_3b_5 + \mu(z_1)\mu(z_2)\mu(z_5) \otimes b_3b_4 + \mu(z_1)\mu(z_3)\mu(z_4) \otimes b_2b_5 + \\
&\mu(z_1)\mu(z_3)\mu(z_5) \otimes b_2b_4 + \mu(z_2)\mu(z_3)\mu(z_5) \otimes b_1b_4 + \mu(z_2)\mu(z_3)\mu(z_4) \otimes b_1b_5 + \\
&\mu(z_3)\mu(z_4)\mu(z_5) \otimes b_1b_2 + \mu(z_2)\mu(z_4)\mu(z_5) \otimes b_1b_3 + \mu(z_1)\mu(z_4)\mu(z_5) \otimes b_2b_3] + \\
&[\mu(z_1)\mu(z_2) \otimes b_3b_4b_5 + \mu(z_1)\mu(z_3) \otimes b_2b_4b_5 + \mu(z_1)\mu(z_4) \otimes b_2b_3b_5 + \mu(z_1)\mu(z_5) \otimes b_2b_3b_4 + \\
&\mu(z_2)\mu(z_3) \otimes b_1b_4b_5 + \mu(z_3)\mu(z_4) \otimes b_1b_2b_5 + \mu(z_4)\mu(z_5) \otimes b_1b_2b_3 + \mu(z_3)\mu(z_5) \otimes b_1b_2b_4 + \\
&\mu(z_2)\mu(z_5) \otimes b_1b_3b_4 + \mu(z_2)\mu(z_4) \otimes b_1b_3b_5] + [\mu(z_1) \otimes b_2b_3b_4b_5 + \mu(z_2) \otimes b_1b_3b_4b_5 + \\
&\mu(z_3) \otimes b_1b_2b_4b_5 + \mu(z_4) \otimes b_1b_2b_3b_5 + \mu(z_5) \otimes b_1b_2b_3b_4] + 1 \otimes b_1b_2b_3b_4b_5 .
\end{aligned}$$

Since $\dim \bar{\mathcal{M}}_{k-2} = 2 \cdot 9 - 16 = 2$, a dimension-counting argument shows that the terms participating in the computation are of the form $\mu(z_{i_1}) \otimes b_{i_2} b_{i_3} b_{i_4} b_{i_5}$, $1 \leq i_j \leq 5$. Therefore, we obtain that

$$\begin{aligned}
\mathcal{F}_9^2(z_1, z_2, z_3, z_4, z_5) &= D_{k-2}(z_1) \mathcal{Q}^{(2)}(\bar{z}_2, \bar{z}_3, \bar{z}_4, \bar{z}_5) + D_{k-2}(z_2) \mathcal{Q}^{(2)}(\bar{z}_1, \bar{z}_3, \bar{z}_4, \bar{z}_5) + \dots \\
&= \sum_{\sigma \in S_5} D_{k-2}(z_{\sigma(1)}) \mathcal{Q}^{(2)}(\bar{z}_{\sigma(2)}, \bar{z}_{\sigma(3)}, \bar{z}_{\sigma(4)}, \bar{z}_{\sigma(5)}) .
\end{aligned}$$

4.3.2 The general case

We let the parameters d and l vary in order to derive the general formula for \mathcal{F}_d^l . Let $2d = 8k - 3(1 + b^+)$ be the dimension of the moduli space \mathcal{M}_k . For any $0 \leq l \leq k$, we consider classes $z_i \in H_2(X; \mathbf{Z})$, $1 \leq i \leq d - 2l$. We aim to compute the pairing

$$\mathcal{F}_d^l(z_1, \dots, z_{d-2l}) = \langle (j_l)^*(\tilde{\mu}(z_1), \dots, \tilde{\mu}(z_{d-2l})), [\bar{\mathcal{M}}_{k-l} \times Sp^l(X)] \rangle .$$

Let $j_l : \bar{\mathcal{M}}_{k-l} \times Sp^l(X) \hookrightarrow \mathcal{IM}_{k-l} \times Sp^l(X) \hookrightarrow \mathcal{IM}_k$, $0 \leq l \leq k$, be the natural inclusion and $j_l^* : H^{2d-4l}(\mathcal{IM}_k) \longrightarrow H^{2d-4l}(\bar{\mathcal{M}}_{k-l} \times Sp^l(X))$ be the induced map in cohomology. Since $\dim \bar{\mathcal{M}}_{k-l} = 2d - 8l$, we set $d \geq 4l$ to ensure that $\bar{\mathcal{M}}_{k-l}$ is non-empty. We first need to determine the class $(j_l)^*(\tilde{\mu}(z_1) \dots \tilde{\mu}(z_{d-2l}))$. The discussion of Section 3.4 ensures that

$$(j_l)^*(\tilde{\mu}(z_1), \dots, \tilde{\mu}(z_{d-2l})) = [\mu(z_1) \otimes 1 + 1 \otimes b_1] \cdot \dots \cdot [\mu(z_{d-2l}) \otimes 1 + 1 \otimes b_{d-2l}] ,$$

where b_i is the *symmetric sum* of copies of the Poincare dual of z_i , $1 \leq i \leq d-2l$. Namely, let $\pi_j : X \times X \rightarrow X$, $j = 1, 2, \dots, l$ be the natural j -th projection and $\pi_j^* : H^2(X) \rightarrow H^2(\underbrace{X \times \dots \times X}_l)$ be the induced map in cohomology. For $z_i \in H^2(X; \mathbb{Z})$, let \bar{z}_i denote the Poincare dual of z_i , $1 \leq i \leq d-2l$. Then, we define the class $b_i \in H^2(Sp^l(X))$ by

$$b_i = [\pi_1^*(\bar{z}_i) + \dots + \pi_l^*(\bar{z}_i)] \equiv [\bar{z}_i \otimes 1 \otimes 1 \otimes \dots \otimes 1 + 1 \otimes \bar{z}_i \otimes 1 \otimes \dots \otimes 1 + \dots + 1 \otimes 1 \otimes \dots \otimes 1 \otimes \bar{z}_i] ,$$

where the brackets $[]$ indicate the 'symmetrization' of $\pi_1^*(\bar{z}_i) + \dots + \pi_l^*(\bar{z}_i)$.^{†4} A straightforward computation gives that

$$\begin{aligned} (j_l)^*(\tilde{\mu}(z_1), \dots, \tilde{\mu}(z_{d-2l})) &= \mu(z_1) \dots \mu(z_{d-2l}) \otimes 1 + [\mu(z_1)\mu(z_2) \dots \mu(z_{d-2l-1}) \otimes b_{d-2l} + \\ &\mu(z_1)\mu(z_2) \dots \mu(z_{d-2l-2})\mu(z_{d-2l}) \otimes b_{d-2l-1} + \mu(z_1)\mu(z_2) \dots \mu(z_{d-2l-3})\mu(z_{d-2l-1})\mu(z_{d-2l}) \\ &\otimes b_{d-2l-2} + \dots] + \dots + [\mu(z_1) \otimes b_2 \dots b_{d-2l-1} b_{d-2l} + \mu(z_2) \otimes b_1 b_3 \dots b_{d-2l-1} b_{d-2l} + \\ &\mu(z_{d-2l}) \otimes b_1 \dots b_{d-2l-1}] + 1 \otimes b_1 \dots b_{d-2l} . \end{aligned}$$

Let now $i_l : Sp^l(X) \rightarrow \mathcal{IM}_k$ be the obvious inclusion. One then checks that

$$\langle i_l^*(\tilde{\mu}(z_1) \dots \tilde{\mu}(z_{2l})), [Sp^l(X)] \rangle = \langle b_1 \dots b_{2l}, [Sp^l(X)] \rangle = Q^{(l)}(\bar{z}_1, \dots, \bar{z}_{2l}) ,$$

where the function $Q^{(l)}$ is defined in terms of the intersection form Q of X by

$$Q^{(l)}(\bar{z}_1, \dots, \bar{z}_{2l}) = \frac{1}{2^l l!} \sum_{\sigma \in S_{2l}} Q(\bar{z}_{\sigma(1)}, \bar{z}_{\sigma(2)}) \dots Q(\bar{z}_{\sigma(d-4l+1)}, \bar{z}_{\sigma(d-2l)}) .$$

We bear in mind that $\dim \bar{\mathcal{M}}_{k-l} = 2d - 8l$ and we evaluate the above class on $\bar{\mathcal{M}}_{k-l} \times Sp^l(X)$ to obtain that

$$\mathcal{F}_d^l(z_1, \dots, z_{d-2l}) = \begin{cases} \sum_{\sigma \in S_{d-2l}} D_{k-l}(z_{\sigma(1)}, \dots, z_{\sigma(d-4l)}) \cdot Q^{(l)}(\bar{z}_{\sigma(d-4l+1)}, \dots, \bar{z}_{\sigma(d-2l)}), & \text{if } d > 4l \\ Q^{(l)}(\bar{z}_1, \dots, \bar{z}_{2l}), & \text{if } d = 4l . \end{cases}$$

^{†4} We note that the class $\bar{z}_i \otimes 1 \otimes 1 \otimes \dots \otimes 1 + 1 \otimes \bar{z}_i \otimes 1 \otimes \dots \otimes 1 + \dots + 1 \otimes 1 \otimes \dots \otimes 1 \otimes \bar{z}_i \in H^2(X \times \dots \times X)$ is invariant under the action of the symmetric group S_l of order l . If $q : X \times \dots \times X \rightarrow Sp^l(X)$ is the quotient map and $H^2(X \times \dots \times X)^{S_l}$ consists of all S_l -invariant 2-cohomology classes of $X \times \dots \times X$, then, $q^* : H^2(X \times \dots \times X)^{S_l} \rightarrow H^2(Sp^l(X))$ is an isomorphism. By $b = [\bar{z}_i \otimes 1 \otimes \dots \otimes 1 + \dots + 1 \otimes \bar{z}_i \otimes \dots \otimes 1] \in H^2(Sp^l(X))$ we denote the class which is such that $q^*(b) = \bar{z}_i \otimes 1 \otimes \dots \otimes 1 + \dots + 1 \otimes \bar{z}_i \otimes \dots \otimes 1$.

4.3.3 Computational variations

We derive variations of the formulae of §4.3.2 by evaluating the cohomology of \mathcal{IM}_k on homologies of $Sp^l(X)$ other than the top-dimensional one. More precisely, by weighting the cohomology of \mathcal{IM}_k on the top homology of $Sp^l(X)$, one does not exploit a great deal of homological information that the latter carries. Motivated by this observation we proceed to evaluate the class $(j_l)^*(\tilde{\mu}(z_1)\dots\tilde{\mu}(z_{d-3l}))$ on $[\bar{\mathcal{M}}_{k-l}] \times [Sp_{2l}]$, $0 \leq l \leq k$, where $[Sp_{2l}]$ is a ‘suitably chosen’ $2l$ -homology class of $Sp^l(X)$. We deal with the special case $l = 2$, whilst the extension of the method to the general case is straightforward.

We set $l = 2$ and for $z_1, z_2 \in H_2(X)$ we consider the class $Sp_4 = z_1 \otimes z_2 + z_2 \otimes z_1 \in H_4(X \times X)$. As in Section 4.3.2, let $[z_1 \otimes z_2 + z_2 \otimes z_1]$ be the isomorphic image of Sp_4 into $H_4(Sp^2(X))$.

(a). Let us fix $d = 8$ and pick classes $z_i \in H_2(X)$, $i = 1, 2$. We recall that $(j_2)^*(\tilde{\mu}(z_1)\tilde{\mu}(z_2)) = [\mu(z_1) \otimes 1 + 1 \otimes b_1] \cdot [\mu(z_2) \otimes 1 + 1 \otimes b_2]$. The only term that participates in the computation (due to dimension consistency) is $1 \otimes b_1 b_2$. Let also $q : X \times X \rightarrow Sp^2(X)$ be the quotient map. We obtain that

$$\begin{aligned} < b_1 b_2, q_*([Sp_4]) > = < q^*(b_1 b_2), [Sp_4] > = < (\bar{z}_1 \otimes 1 + 1 \otimes \bar{z}_1)(\bar{z}_2 \otimes 1 + 1 \otimes \bar{z}_2), [Sp_4] > = \\ & 2! < \bar{z}_1 \otimes \bar{z}_2 + \bar{z}_2 \otimes \bar{z}_1, [z_1 \otimes z_2 + z_2 \otimes z_1] > = 2! [< z_1, \bar{z}_1 > < z_2, \bar{z}_2 > \\ & + < z_1, \bar{z}_2 > < z_2, \bar{z}_1 >] = 2! [Q(z_1, z_1) \cdot Q(z_2, z_2) + Q(z_1, z_2)^2]. \end{aligned}$$

Therefore, for $d = 8$ we have that

$$< (j_2)^*(\tilde{\mu}(z_1)\tilde{\mu}(z_2)), [\bar{\mathcal{M}}_{k-2}] \times [Sp_4] > = 2! [Q(z_1, z_1) \cdot Q(z_2, z_2) + Q(z_1, z_2)^2].$$

(b). We set $d = 9$ and pick classes $z_i \in H_2(X)$, $i = 1, 2, 3$ (since $d - 3l = 9 - 6 = 3$). We then obtain that

$$(j_2)^*(\tilde{\mu}(z_1)\tilde{\mu}(z_2)\tilde{\mu}(z_3)) = [\mu(z_1) \otimes 1 + 1 \otimes b_1] \cdot [\mu(z_2) \otimes 1 + 1 \otimes b_2] \cdot [\mu(z_3) \otimes 1 + 1 \otimes b_3].$$

Since $\dim M_{k-2} = 2d - 8l = 2$, the terms which participate in the computation are of the form $\mu_{i_1} \otimes b_{i_2} b_{i_3}$. We easily check that

$$< b_2 b_3, q_*([Sp_4]) > = 2! (Q(z_1, z_2) \cdot Q(z_2, z_3) + Q(z_2, z_2) \cdot Q(z_1, z_3))$$

$$\langle b_1 b_3, q_*([Sp_4]) \rangle = 2!(Q(z_1, z_1) \cdot Q(z_2, z_3) + Q(z_1, z_2) \cdot Q(z_1, z_3))$$

$$\langle b_1 b_2, q_*([Sp_4]) \rangle = 2!(Q(z_1, z_2)^2 + Q(z_1, z_1) \cdot Q(z_2, z_2)) .$$

Therefore, for $d = 9$ we obtain that:

$$\begin{aligned} (j_2)^*(\tilde{\mu}(z_1)\tilde{\mu}(z_2)\tilde{\mu}(z_3)), [\bar{\mathcal{M}}_{k-2}] \times [Sp_4] &= 2![D_{k-2}(z_1)(Q(z_1, z_2) \cdot Q(z_2, z_3) + \\ &Q(z_2, z_2) \cdot Q(z_1, z_3)) + D_{k-2}(z_2)(Q(z_1, z_1) \cdot Q(z_2, z_3) + Q(z_1, z_2) \cdot Q(z_1, z_3)) + \\ &D_{k-2}(z_3)(Q(z_1, z_2)^2 + Q(z_1, z_1) \cdot Q(z_2, z_2))] . \dagger^5 \end{aligned}$$

By varying the values of d and applying the same methodology, we produce a family of (not necessarily distinct) formulae which are given in terms of Donaldson polynomials of certain degree and the intersection form of X .

Remark 4.3.2 (a) We can repeat the above argument for different values of l . For instance, for $l = 3$ we would proceed to weight the class $(j_2)^*(\tilde{\mu}(z_1) \dots \tilde{\mu}(z_{d-3l}))$ on $[\bar{\mathcal{M}}_{k-3}] \times [Sp_6]$, where Sp_6 is the symmetrised version of the class $z_1 \otimes z_2 \otimes z_3$, namely

$$Sp_6 = z_1 \otimes z_2 \otimes z_3 + z_2 \otimes z_1 \otimes z_3 + z_3 \otimes z_1 \otimes z_2 + z_1 \otimes z_3 \otimes z_2 + z_2 \otimes z_3 \otimes z_1 + z_3 \otimes z_2 \otimes z_1 .$$

(b) A particular feature of the formulae derived above is the evaluation of the intersection form of the underlying 4-manifold X on 2-tuples of *identical* homology classes.

4.4 An informal discussion

An immediate task emerged by the computations of Section 4.3 is the utilization of the family of the formulae obtained. One could anticipate that in some cases, certain products of $\tilde{\mu}$ -classes of \mathcal{IM}_k vanish on certain boundary components, establishing in this way universal relations that Donaldson polynomials satisfy. Such situations would essentially depend upon suitable choices of the parameters involved, such as l and d . The following conjecture puts our speculation into formal terms.

⁵† Apparently, one can as well evaluate on either $[z_2 \otimes z_3 + z_3 \otimes z_2]$ or $[z_1 \otimes z_3 + z_3 \otimes z_1]$. This calculation will obviously yield the same type of formulae after rotating z_1, z_2, z_3 .

Conjecture 4.4.1 *For fixed d , there exists l such that the pairing of the product $\tilde{\mu}(z_1) \dots \tilde{\mu}(z_{d-2l}) \in H^{2d-4l}(\mathcal{IM}_k)$ with the fundamental class of the boundary component $\bar{\mathcal{M}}_{k-l} \times Sp^l(X)$, $0 \leq l \leq k$, vanishes.*

Let us interpret Conjecture 4.4.1 in topological as well as in geometrical language.

(a). Let \mathcal{C} be the set of boundary points which we ‘add’ on the ASD moduli space \mathcal{M}_k in order to compactify it, namely $\mathcal{IM}_k \setminus \mathcal{C} = \mathcal{M}_k$. We consider the (trivial) map π of pairs $(\mathcal{IM}_k, \emptyset) \hookrightarrow (\mathcal{IM}_k, \mathcal{C})$ and also the inclusion $\mathcal{C} \hookrightarrow \mathcal{IM}_k$. The associated exact cohomology sequence, in dimension N say, is then given by

$$H^N(\mathcal{IM}_k, \mathcal{C}) \xrightarrow{\pi^*} H^N(\mathcal{IM}_k) \xrightarrow{i^*} H^N(\mathcal{C}).$$

By *Lefschetz duality*⁶ we have that $H^N(\mathcal{IM}_k, \mathcal{C}) \cong H_{2d-N}(\mathcal{M}_k)$. According to the *Atiyah-Jones conjecture* [5], for large enough k , we have that $H_{2d-N}(\mathcal{M}_k) \cong H_{2d-N}(\tilde{B}_k)$, where \tilde{B}_k is the moduli space of framed connections [8]. Let us confine our discussion to the case $N = 2d - 4l$, $0 \leq l \leq k$. Then, our exact sequence becomes

$$H_{4l}(\tilde{B}_k) \cong H_{4l}(\mathcal{M}_k) \cong H^{2d-4l}(\mathcal{IM}_k, \mathcal{C}) \xrightarrow{\pi^*} H^{2d-4l}(\mathcal{IM}_k) \xrightarrow{i^*} H^{2d-4l}(\mathcal{C}).$$

We pick $z_1, \dots, z_{d-2l} \in H_2(X)$ and consider the class $\tilde{\mu}(z_1) \dots \tilde{\mu}(z_{d-2l}) \in H^{2d-4l}(\mathcal{IM}_k)$. The question that the conjecture poses, is for which value(s) of l , there exists a class $Z \in H_{4l}(\tilde{B}_k)$ such that $\pi^*(Z) = \tilde{\mu}(z_1) \dots \tilde{\mu}(z_{d-2l})$. For those values of l and using the exactness of the sequence we would have that $i^*(\tilde{\mu}(z_1) \dots \tilde{\mu}(z_{d-2l})) = 0$, which would eventually imply that $\mathcal{F}_d^l(z_1, \dots, z_{d-2l}) = 0$.

(b). We present Conjecture 4.5.1 in geometric language. To do so, we employ the representation of a class $\bar{\mu}(z) \in H^2(\bar{\mathcal{M}}_k)$ as the Poincare-dual of a codimension 2 submanifold V_z of \mathcal{M}_k . For a detailed description of V_z we refer the reader to [10], §5.2.2 of [8] or §2 of [11].

Let us just mention that if Σ is an embedded surface in X which represents the 2-homology class z , then let V_z be the zero-set of a transverse section of the determinant

⁶† The Lefschetz duality theorem states that if the pair (M, C) is such that, M is a Hausdorff space, C is closed in M and $M \setminus C$ is an orientable n -manifold, then, for all q there is an isomorphism $H_q(M \setminus C) \cong H^{n-q}(M; C)$.

line bundle \mathcal{L}_Σ (over the moduli space B_Σ of connections over Σ) for a particular family of operators associated to Σ . Let also $r_\Sigma : B_X \rightarrow B_\Sigma$ be the apparent restriction map. The set $V_\Sigma = r_\Sigma^{-1}(V_z)$ is a codimension 2 submanifold of \mathcal{M}_k which is dual to the class $\mu(z)$. An extension of this result to the case of more than one surface is given below.

Theorem 4.4.2 *If $\Sigma_1, \dots, \Sigma_d$ are embedded surfaces, then, the sets V_{z_1}, \dots, V_{z_d} can be chosen such that all the intersections $\mathcal{M}_k \cap V_{z_{i_1}} \cap \dots \cap V_{z_{i_d}}$, $i_1 < \dots < i_d$, are transverse. The intersection is then a smooth submanifold dual to $\mu(z_{i_1}) \dots \mu(z_{i_d})$ in \mathcal{M}_k .*

Therefore, Conjecture 4.4.1 asks for which values of l , the $4l$ -dimensional manifold $V \equiv \mathcal{M}_k \cap V_{z_1} \cap \dots \cap V_{z_{d-2l}}$ does not meet the stratum $\mathcal{M}_{k-l} \times Sp^l(X)$ since the pairing $\mathcal{F}_d^l(z_1, \dots, z_{d-2l}) \equiv \langle (j_l)^*(\bar{\mu}(z_1) \dots \bar{\mu}(z_{d-2l})), [\bar{\mathcal{M}}_{k-l} \times Sp^l(X)] \rangle$ represents the number of the elements of the intersection $(\mathcal{M}_k \cap V_{z_1} \cap \dots \cap V_{z_{d-2l}}) \cap (\mathcal{M}_{k-l} \times Sp^l(X))$ counted by signs.

Remark 4.4.3 Conjecture 4.4.1 seems to be ‘sensitive’ upon changes on the values of $b^+(X)$. More precisely, as pointed out in [10], there is a gap between the situation $b^+(X) \leq 2$ and the situation $b^+(X) \geq 3$ as far as the intersection of V with the lower strata of $\bar{\mathcal{M}}_k$ is concerned.

In the former case, S. Donaldson [10] cut the moduli space in a way to avoid all the intermediate strata - see Lemma 8.3.3 of [8]. The situation that occurs when $b^+(X) \geq 3$ is that V will in general have additional ‘ends’ associated with the intersection of the closure \bar{V} with the other strata.⁷ For example, when $b^+ = 3$ and $k = 4$, the dimension of V is 4 and we can expect \bar{V} to have isolated points of intersection with the 16-dimensional stratum $\mathcal{M}_3 \times X$ inside the 20-dimensional space $\bar{\mathcal{M}}_4$. The study of the case $b^+(X) \geq 3$ is undertaken in §2 of [36] and yields *mod 2* universal constraints for Donaldson polynomials of odd charge on even manifolds.

For a discussion on the use of a *mod 2* version of the computations emerged by Conjecture 4.4.1 in obtaining topological information about spin 4-manifolds with indefinite intersection form as well as the relevance of the setting to the formulae derived in Section 4.3, we refer to §5 of [25].

⁷† See Appendix of [10].

Questions and Observations

We address some questions arisen from this thesis and comment on their potential contribution to gauge-theoretical issues.

(a). Compute the cohomology groups of the space \mathcal{JB}_k constructed in Chapter 2. We recall that by construction the space \mathcal{JB}_k is formed by the ‘gluing’ of the spaces $\mathcal{JB}_{k,q-1}$ and $\mathcal{A}_{k-q} \times_{\mathcal{G}_{k-q}} Sp^q(E_{k-q})$ along the stratum $\mathcal{A}_{k-q} \times_{\mathcal{G}_{k-q}} Sp^q(E_{k-q})_{q-1}$, $1 \leq q \leq k$. This observation suggests that one could consider the associated Mayer-Vietoris exact sequences and also exploit information concerning the cohomology of the individual strata - see §5.1.2 of [8] - in order to carry out the aforementioned computation.

(b). Use the Mayer-Vietoris sequence associated to the stratified space \mathcal{JM}_k constructed in Chapter 1, to give an alternative definition of the $\bar{\mu}$ -map described in Chapter 3. Note that due to the exactness of the Mayer-Vietoris sequence as well as the properties of the $\bar{\mu}$ -map mentioned in Section 3.4, the argument would be essentially converted to proving by induction on q that for $z \in H^2(X; \mathbf{Z})$, the classes $\mu_{k-q-1}(z) \in H^2(\mathcal{JM}_{k-q-1})$ and $\mu^{(k-q)}(z) \oplus q \cdot PD(z) \in H^2(\mathcal{A}_{k-q} \times_{\mathcal{G}_{k-q}} Sp^q(E_{k-q}))$ have the same image into $H^2(\mathcal{A}_{k-q} \times_{\mathcal{G}_{k-q}} Sp^q(E_{k-q})_{q-1})$, $1 \leq q \leq k$.

(c). Note that the space \mathcal{JM}_k with its identification topology τ is compact, in order to form the pairing between its fundamental class and certain products of $\tilde{\mu}$ -classes obtained in (b). Then, the observation that the space (\mathcal{JM}_k, τ) coincides with the ideal moduli space \mathcal{IM}_k with the topology induced by the notion of weak convergence, would lead to a simple definition of the Donaldson polynomials. This fact in conjunction with the study of the multiple-gluing maps \mathcal{F}_l of Chapter 1 on the level of homology, would then lead to the discovery of universal relations among Donaldson polynomials of different degrees.

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